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E L E M E N T S
OF
G E O M E T R Y.
CONTAINING THE
PRINCIPAL PROPOSITIONS
IN THE
FIRST SIX, AND THE ELEVENTH AND
TWELFTH BOOKS
OF
E U C L I D.
WITH NOTES CRITICAL AND EXPLANATORY.
By JOHN BONNYCASTLE,
OF THE ROYAL MILITARY ACADEMY, WOOLWICH.
THE THIRD EDITION.

L O N D O N:
PRINTED FOR J. JOHNSON, NO. 72, ST. PAUL'S
CHURCH-YARD,
BY EYE AND LAW, ST. JOHN'S-SQUARE, CLERKENWELL.

1803.



P R E F A C E.

OF all the works of antiquity which have been transmitted to the present times, none are more universally and deservedly esteemed than the *Elements of Geometry* which go under the name of *EUCLID*. In many other branches of science the moderns have far surpassed their masters; but, after a lapse of more than two thousand years, this performance still maintains its original pre-eminence, and has even acquired additional celebrity from the fruitless attempts which have been made to establish a different system.

It is, however, generally allowed, that the *Elements*, as they now stand, are attended with many difficulties, which greatly retard the progress of learners, on their first entrance upon this study, and prevent them from applying to other branches of knowledge, which, in the present advanced state of the sciences, are equally useful and important. Among other obstacles of this kind

may be mentioned the theory of parallel lines, the doctrine of proportion, and many things in the eleventh and twelfth books, relating to solids, which are usually found extremely embarrassing ; and notwithstanding the numberless efforts which have been made to elucidate and explain them, are still liable to many objections.

On this account, it has been found necessary, in most of our academical institutions, to have recourse to some of the more compendious rudiments of later writers, who, by means of a different arrangement, have endeavoured to new-model the subject, and to render it less complex and elaborate. But the greater part of them are so ill digested that they serve rather to mislead the learner than to afford him any assistance. For, besides being deficient in order and method, some of these authors have treated the subject algebraically ; and others, by introducing a number of exceptionable principles, and a vague unsatisfactory mode of demonstration, have degraded the science, and deprived it of some of its most striking advantages.

It

P R E F A C E.

It is, therefore, the design of the following performance, to obviate these objections, and to render the subject more familiar and perspicuous, without weakening its evidence, or destroying its elegance and simplicity. For this purpose, many propositions in *EUCLID*, which are of little or no use in their application, and were only introduced into the *Elements* as necessary links in the chain of reasoning, are here omitted; and others substituted in their place, which are equally conducive to that end, and at the same time more useful and concise. By this means all the most essential principles of the science have been brought into a shorter compass, and the demonstrations, which lead to its sublimer truths, so continued, as to render their connection as obvious and comprehensive as possible.

Great care has also been taken to preserve that methodical precision and rigour of proof, which, in treating of this subject, are requisites of nearly equal importance with the science itself. For independently of its other advantages, Geometry has always been considered as an excellent logic, which in forming

ing the mind, and establishing a habit of close thinking and just reasoning, in every enquiry after truth, is far superior to all the dialectical principles that have yet been invented; the simplicity of its first principles; the clearness and certainty of its demonstrations; the regular concatenation of its parts; and the universality of its application being such as no other subject can boast.

For these reasons, it was judged necessary to adhere as closely as possible to the plan of the original Elements; this being, in many respects, much more natural and judicious than any of those which have since been proposed by other writers. But as the work was rather designed as a regular Institution of the most useful principles of the science, than a strict abridgment of EUCLID, some alterations have been made, both in the arrangement of the propositions and the mode of demonstration; the latter of which, in particular, it is presumed, will be found considerably improved, being here delivered in a more convenient form, and rendered as clear and explicit as the nature of the subject would admit.

In

In the first six books, every thing has been demonstrated with a scrupulous accuracy; and it was at first designed that the same method should have been observed throughout; but this, in treating of the solids, was found incompatible with the plan of the work, it being here scarcely possible to follow the strict principles of EUCLID without becoming prolix and obscure. It was therefore thought proper, in this part of the performance, to adopt a mode of proof, which though not geometrically exact, is far more perspicuous than the former, and equally satisfactory and convincing to the mind; especially in the way it is here given, which is something less exceptionable than that of CAVALERIUS, by whom it was first introduced.

Many other particulars might be mentioned, in which this performance will be found to differ from most others of the like nature; but as they consist chiefly of improvements and emendations which are too obvious to escape the notice of the reader, any further account of them would be unnecessary. It is sufficient to observe that much time and

attention have been bestowed upon the work ; and that nothing which was judged essential to the science, or useful in facilitating its attainment, has been omitted. The acknowledged intricacy of some propositions in the fifth and sixth books, made it necessary to abridge that part of the subject more considerably than the former ; but it is conceived that what is here given will be fully sufficient to answer all the purposes of the learner.

To avoid critical objections were a vain endeavour : they may be made against every system of Geometry now extant ; and to *EUCLID* as well as to other writers. Of this abundant proofs are given by the Commentators ; and in the Notes at the end of the present work, where many things of this kind are pointed out which have hitherto escaped notice. These were added chiefly for the information of young students, and ought to be carefully consulted by those who wish to obtain a just idea of the science, and the principles upon which it is founded.

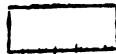
THE
ELEMENTS
OF
GEOMETRY.
BOOK I.

DEFINITIONS.

1. A Solid is that which has length, breadth and thickness.



2. A Superficies is one of the bounds of a solid, and has length and breadth without thickness.



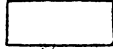
3. A Line is one of the bounds of a superficies, and has length without breadth or thickness.



4. A Point is one of the extremities of a line, and has neither length, breadth, nor thickness.

5. A right line is that which has all its parts lying in the same direction.

6. A plane superficies is that which is every where perfectly flat and even.



7. A plain rectilineal angle is the inclination or opening of two right lines which meet in a point.



8. One right line is said to be perpendicular to another, when it makes the angles on both sides of it equal to each other.



9. A right angle is that which is made by two right lines that are perpendicular to each other.



10. An obtuse angle is that which is greater than a right angle.



11. An acute angle is that which is less than a right angle.



12. A figure is that which is inclosed by one or more boundaries.

13. A circle is a plane figure, contained by one line, called the circumference, which is every where equally distant from a point within the figure, called the centre.



14. Rectilineal figures are those which are contained by right lines.

15. All plane figures, bounded by three right lines, are called triangles.

16. An equilateral triangle, is that which has all its sides equal to each other.



17. An isosceles triangle, is that which has only two of its sides equal to each other.



18. A right-angled triangle, is that which has one right angle; the side which is opposite to the right angle being called the hypotenuse.

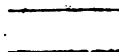


4 ELEMENTS OF GEOMETRY.

19. An obtuse-angled triangle, is that which has one obtuse angle.

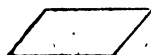


20. Parallel right lines are such as are in the same plane, and which, being produced ever so far both ways, will never meet.

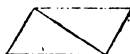


21. Every plane figure, bounded by four right lines, is called a quadrangle, or quadrilateral.

22. A parallelogram, is a quadrangle whose opposite sides are parallel.



23. The diagonal of a quadrangle, is a right line joining any two of its opposite angles.



24. The base of any figure is that side upon which it is supposed to stand; and the vertical angle is that which is opposite to the base.



NOTE, When an angle is expressed by means of three letters, the one which stands at the angular point, must always be placed in the middle.

BOOK THE FIRST.

3

POSTULATES.

1. Let it be granted that a right line may be drawn from any one given point to another.
2. That a terminated right line, may be produced to any length in a right line.
3. That a circle may be described from any point as a centre, at any distance from that centre.
4. And that a right line, which meets one of two parallel right lines, may be produced till it meets the other.

A X I O M S.

1. Things which are equal to the same thing are equal to each other.
2. If equals be added to equals the wholes will be equal.
3. If equals be taken from equals the remainders will be equal.
4. If equals be added to unequals the wholes will be unequal.

B 3

5. If

5. If equals be taken from unequals the remainders will be unequal.

6. Things which are double of the same thing are equal to each other.

7. Things which are halves of the same thing are equal to each other.

8. The whole is equal to all its parts taken together.

9. Magnitudes which coincide, or fill the same space, are equal to each other.

REMARKS.

A PROPOSITION, is something which is either proposed to be done, or to be demonstrated.

A PROBLEM, is something which is proposed to be done.

A THEOREM, is something which is proposed to be demonstrated.

A LEMMA, is something which is previously demonstrated, in order to render what follows more easy.

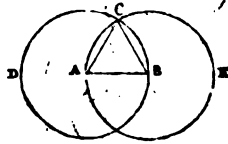
A COROLLARY, is a consequent truth, gained from some preceding truth, or demonstration.

A SCHOLIUM, is a remark or observation made upon something going before it.

P R O.

PROPOSITION I. (PROBLEM.)

ON a given finite right line to describe an equilateral triangle.



Let AB be the given right line; it is required to describe an equilateral triangle upon it.

From the point A, at the distance AB, describe the circle BCD (*Def. 3.*)

And from the point B, at the distance BA, describe the circle ACE (*Def. 3.*)

Then, because the two circles pass through each other's centres, they will cut each other.

And, if the right lines CA, CB be drawn from the point of intersection C, ABC will be the equilateral triangle required.

For, since A is the centre of the circle BCD, AC is equal to AB (*Def. 13.*)

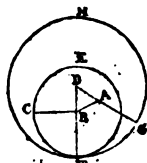
And, because B is the centre of the circle ACE, BC is also equal to AB (*Def. 13.*)

But things which are equal to the same thing are equal to each other (*Ax. 1.*); therefore AC is equal to CB.

And, since AC, CB are equal to each other, as well as to AB, the triangle, ABC is equilateral; and it is described upon the right line AB, as was to be done.

PROP. II. PROBLEM.

From a given point to draw a right line equal to a given finite right line.



Let A be the given point, and BC the given right line; it is required to draw a right line from the point A, that shall be equal to BC.

Join the points A, B, (*Pos. 1.*); and upon BA describe the equilateral triangle BAD (*Prop. 1.*)

From the point B, at the distance BC, describe the circle CEF (*Pos. 3.*) cutting DE produced in F.

And from the point D, at the distance DF, describe the circle FHG (*Pos. 3.*); then, if DA be produced to G, AG will be equal to BC, as was required.

For, since B is the centre of the circle CEF, BC is equal to BF (*Def. 13.*)

And, because D is the centre of the circle FHG, DG is equal to DF (*Def. 13.*)

But the part DA is also equal to the part DB (*Def. 16.*), whence the remainder AG will be equal to the remainder BF (*Ax. 3.*)

And since AG, BC have been each proved to be equal to BF, AG will also be equal to BC (*Ax. 1.*)

A right line AG, has, therefore, been drawn from the point A, equal to the right line BC, as was to be done.

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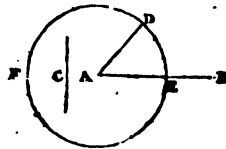
BOOK THE FIRST.

9

SCHOLIUM. When the point A is at one of the extremities B, of the given line BC, any right line, drawn from that point to the circumference of the circle CEF, will be the one required.

PROP. III. PROBLEM.

From the greater of two given right lines, to cut off a part equal to the less.



Let AB and c be the two given right lines; it is required to cut off a part from AB, the greater, equal to c the less.

From the point A draw the right line AD equal to c (*Prop. 2.*); and from the centre A, at the distance AD, describe the circle DEF (*Def. 3.*) cutting AB in E, and AE will be equal to c as was required.

For, since A is the centre of the circle EDF, AE will be equal to AD (*Def. 13.*)

But c is equal to AD, by construction; therefore AE will also be equal to c (*Ax. 1.*)

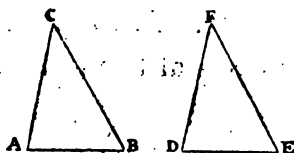
Whence, from AB, the greater of the two given lines, there has been taken a part equal to c the less, which was to be done.

SCHOLIUM. When the two given lines are so situated, that one of the extremities of c falls in the point A, the former part of the construction becomes unnecessary.

PROP.

P R O P. IV. THEOREM.

If two sides and the included angle of one triangle, be equal to two sides and the included angle of another, each to each, the triangles will be equal in all respects.



Let ABC , DEF be two triangles, having CA equal to FD , CB to FE , and the angle C to the angle F ; then will the two triangles be equal in all respects.

For conceive the triangle ABC to be applied to the triangle DEF , so that the point C may coincide with the point F , and the side CA with the side FD .

Then, because CA coincides with FD , and the angle C is equal to the angle F (*by Hyp.*), the side CB will also coincide with the side FE .

And, since CA is equal to FD , and CB to FE (*by Hyp.*), the point A will fall upon the point D , and the point B upon the point E .

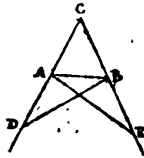
But right lines, which have the same extremities, must coincide, or otherwise their parts would not lie in the same direction, which is absurd (*Def. 5.*); therefore AB falls upon, and is equal to DE .

And, because the triangle ABC coincides with the triangle DEF , the angle A will be equal to the angle D , the angle B to the angle E , and the two triangles will be equal in all respects (*Ax. 9.*) Q. E. D.

P R O P.

P R O P. V. THEOREM.

The angles at the base of an isosceles triangle are equal to each other.



Let ABC be an isosceles triangle, having the side CA equal to the side CB ; then will the angle CAB be equal to the angle CBA .

For, in CA and CB produced, take any two equal parts CD , CE (*Prop. 3.*), and join the points AE , BD :

Then, because the two sides CA , CE of the triangle CAE , are equal to the two sides CB , CD of the triangle CBD , and the angle C is common, the side AE will also be equal to the side BD , the angle CAE to the angle CBD , and the angle D to the angle E (*Prop. 4.*)

And since the whole CD is equal to the whole CE (*by Const.*), and the part CA to the part CB (*by Hyp.*), the remaining part AD will also be equal to the remaining part BE (*Ax. 3.*)

Hence, the two sides DA , DB , of the triangle DAB , being equal to the two sides EB , EA of the triangle EBA , and the angle D to the angle E , the angle ABD will also be equal to the angle BAE (*Prop. 4.*)

And if from the equal angles CAE , CBD , there be taken the equal angles BAE , ABD , the remaining angle

CAB

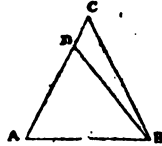
12 ELEMENTS OF GEOMETRY.

CAB will be equal to the remaining angle CBA (*Ax. 3.*)
Q. E. D.

COROLLARY. Every equilateral triangle is also equiangular.

P R O P. VI. THEOREM.

If two angles of a triangle be equal to each other, the sides which are opposite to them will also be equal.



Let ABC be a triangle, having the angle CAB equal to the angle CBA; then will the side CA be equal to the side CB.

For if CA be not equal to CB, one of them must be greater than the other; let CA be the greater, and make AD equal to BC (*Prop. 3.*) and join BD.

Then, because the two sides AD, AB, of the triangle ADB, are equal to the two sides BC, BA, of the triangle ACB, and the angle DAB is equal to the angle CBA (*by Hyp.*), the triangle ADB will be equal to the triangle ACB (*Prop. 4.*), the less to the greater, which is absurd.

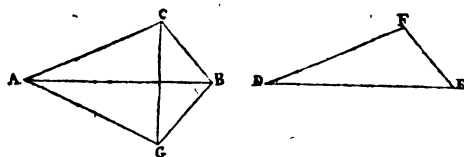
The side CA, therefore, cannot be greater than the side CB; and, in the same manner, it may be shewn that it cannot be less; consequently they are equal to each other. Q. E. D.

COROL. Every equiangular triangle is also equilateral.

P R O P.

PROP. VII. THEOREM.

If the three sides of one triangle be equal to the three sides of another, each to each, the angles which are opposite to the equal sides will also be equal.



Let ABC , DEF be two triangles, having the side AB equal to the side DE , AC to DF , and BC to EF ; then will the angle ACB be equal to the angle DFE , BAC to EDF , and ABC to DEF .

For, let the triangle DEF be applied to the triangle ACB , so that their longest sides, DE , AB , may coincide, and the point F fall at C ; and join CG .

Then, since the side AC is equal to the side DF , or AC (*by Hyp.*), the angle ACG will be equal to the angle AGC (*Prop. 5.*).

And, because the side BC is equal to the side EF , or BC (*by Hyp.*), the angle BCG will be equal to the angle BGC (*Prop. 5.*).

But since the angles ACG , BCG are equal to the angles AGC , BGC , each to each, the whole angle ACB will be equal to the whole angle AGB (*Ax. 8.*)

And, because AC is equal to AG , BC to BG , and the angle ACB to the angle AGB , the angle CAB , will, also, be

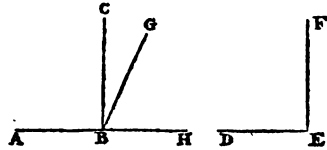
14 ELEMENTS OF GEOMETRY.

be equal to the angle GAB , and the angle ABC to the angle ABG (*Prop. 4.*)

But the triangles AGB , DFE , are identical; consequently the angles of the triangle DFE will, also, be equal to the corresponding angles of the triangle ACB .
Q. E. D.

P R O P. VIII. THEOREM.

All right angles are equal to each other.



Let ABC , DEF be each of them right angles; then will ABC be equal to DEF .

For conceive the angle DEF to be applied to the angle ABC , so that the point E may coincide with the point B , and the line ED with the line BA .

And if EF does not coincide with BC , let it fall, if possible, without the angle ABC , in the direction BG ; and produce AB to H .

Then, because the angles ABC , ABG are right angles (*by Hyp.*), the lines CB , GB will be each perpendicular to AH (*Def. 8.9.*)

And, since a right line which is perpendicular to another right line, makes the angles on each side of it equal (*Def. 8.*), the angle CBA will be equal to the angle CBH , and the angle GBA to the angle GBH .

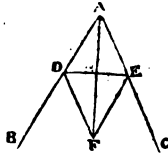
But the angle GBA is greater than the angle CBA , or its equal CBH ; consequently the angle GBH is also greater than
than

than the angle CBH ; that is, a part is greater than the whole, which is absurd.

The line EF , therefore, does not fall without the angle ABC : and in the same manner it may be shewn that it does not fall within it; consequently EF and BC will coincide, and the angle DEF be equal to the angle ABC , as was to be shewn.

P R O P. IX. PROBLEM.

To bisect a given rectilineal angle, that is, to divide it into two equal parts.



Let BAC be the given rectilineal angle; it is required to divide it into two equal parts.

Take any point D in AB , and from AC cut off AE equal to AD (*Prop. 3.*), and join DE .

Upon DE describe the equilateral triangle DEF (*Prop. 1.*), and join AF ; then will AF bisect the angle BAC , as was required.

For AD is equal to AE , by construction; DF is also equal to FE (*Def. 16.*), and AF is common to each of the triangles AFD , AFE .

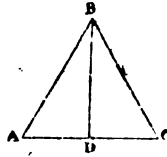
But when the three sides of one triangle are equal to the three sides of another, each to each, the angles which are opposite to the equal sides are, also, equal (*Prop. 7.*)

The

The side DF , therefore, being equal to the side FE , the angle DAF will be equal to the angle FAE ; and consequently the angle BAC is bisected by the right line AF , as was to be done.

P R O P. X. PROBLEM.

To bisect a given finite right line, that is, to divide it into two equal parts.



Let AC be the given right line; it is required to divide it into two equal parts.

Upon AC describe the equilateral triangle ACB (*Prop. 1.*), and bisect the angle ABC by the right line BD (*Prop. 9.*); then will AC be divided into two equal parts at the point D , as was required.

For AB is equal to BC (*Def. 16.*), BD is common to each of the triangles ADB , CDB , and the angle ABD is equal to the angle CBD (*by Const.*)

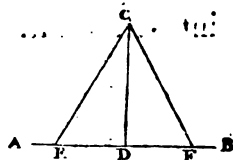
But when two sides and the included angle of one triangle, are equal to two sides and the included angle of another, each to each, their bases will also be equal (*Prop. 4.*)

The base AD is, therefore, equal to the base DC ; and, consequently, the right line AC is bisected in the point D , as was to be done.

P R O P.

PROP. XI. PROBLEM.

At a given point, in a given right line, to erect a perpendicular.



Let AB be the given right line, and D the given point in it; it is required to draw a right line, from the point D , that shall be perpendicular to AB .

Take any point E , in AB , and make DF equal to DE (*Prop. 3.*), and upon EF describe the equilateral triangle ECF (*Prop. 1.*)

Join the points D, C ; and the right line CD will be perpendicular to AB , as was required.

For CE is equal to CF (*Def. 16.*), ED to DF (*by Const.*) and CD is common to each of the triangles ECD, FCD .

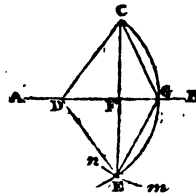
The three sides of the triangle ECD being, therefore, equal to the three sides of the triangle FCD , each to each, the angle EDC will, also, be equal to the angle FDC (*Prop. 7.*)

But one right line is perpendicular to another when the angles on both sides of it are equal (*Def. 8.*); therefore CD is perpendicular to AB ; and it is drawn from the point D as was to be done.

SCHOLIUM. If the given point be at, or near, the end of AB ; the line must be produced.

P R O P. XII. PROBLEM.

To draw a right line perpendicular to a given right line, of an unlimited length, from a given point without it.



Let AB be the given right line, and c the given point; it is required to draw a right line from the point c , that shall be perpendicular to AB .

Take any point D , in AB , and from that point, with the distance DC , describe the circle CDE , cutting AB in E .

Join GC , and from the point G , with the distance GC , describe the circle nEm , cutting the former in E .

Through the points C , E draw the right line CFE , cutting AB in F , and CF will be perpendicular to AB , as was required.

For, join the points D , C , D , E , and G , E :

Then, because DC is equal to DE , GC to GE (*Def. 13.*) and DG common to each of the triangles DCG , DEG , the angle CDG will be equal to the angle CDE (*Prop. 7.*)

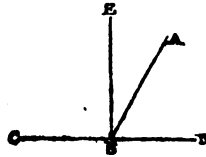
And, since DC is equal to DE , DF common to each of the triangles DCF , DEF , and the angle CDG equal
to

to the angle CDE , the angle DFC will also be equal to the angle DFE (*Prop. 4.*)

But one line is perpendicular to another when the angles on both sides of it are equal (*Def. 8*); therefore CF is perpendicular to AB ; and it is drawn from the point C , as was to be done.

P R O P. XIII. THEOREM.

The angles which one right line makes with another, on the same side of it, are together equal to two right angles.



Let the right line AB fall upon the right line CD ; then will the angles ABC , ABD , taken together, be equal to two right angles.

For if the angles ABC , ABD be equal to each other, they will be, each of them, right angles (*Def. 8 and 9.*)

But if they be unequal, let EB be drawn, from the point B , perpendicular to CD (*Prop. 11.*)

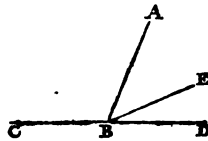
Then, since the angles EBC , EBD are right angles (*Def. 8.*), and the angle EBD is equal to the angles EBA , ABD (*Ax. 8.*), the angles EBC , EBA and ABD will be equal to two right angles.

But the angles EBC , EBA are, together, equal to the angle ABC (*Ax. 8.*); consequently the angles ABC , ABD are, also, equal to two right angles. *Q. E. D.*

COROLL. All the angles which can be made, at any point B , on the same side of the right line CD , are, together, equal to two right angles.

P R O P. XIV. THEOREM.

If a right line meet two other right lines, in the same point, and make the angles on each side of it together equal to two right angles, those lines will form one continued right line.



Let the right line AB meet the two right lines CB , BD , at the point B , and make the angles ABC , ABD together equal to two right angles, then will BD be in the same right line with CB .

For, if it be not, let some other line BE be in the same right line with CB .

Then, because the right line AB falls upon the right line CBE , the angles ABC , ABE , taken together, are equal to two right angles (*Prop. 13.*)

But the angles ABC , ABD are also equal to two right angles (*by Hyp.*); consequently the angles ABC , ABE are equal to the angles ABC , ABD .

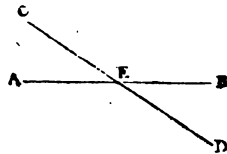
And,

And, if the angle ABC , which is common, be taken away, the remaining angle ABE will be equal to the remaining angle ABD ; the less to the greater, which is absurd.

The line BE , therefore, is not in the same right line with CB ; and the same may be proved of any other line but BD ; consequently CBD is one continued right line, as was to be shewn.

P R O P. XV. THEOREM.

If two right lines intersect each other, the opposite angles will be equal.



Let the two right lines AB , CD intersect each other in the point E ; then will the angle AEC be equal to the angle DEB , and the angle AED to the angle CEB .

For, since the right line CE falls upon the right line AB , the angles AEC , CEB , taken together, are equal to two right angles (*Prop. 13.*)

And, because the right line BE falls upon the right line CD , the angles BED , CEB , taken together, are also equal to two right angles (*Prop. 13.*)

The angles AEC , CEB , taken together, are, therefore, equal to the angles BED , CEB taken together (*Ax. 1.*)

C 3

And,

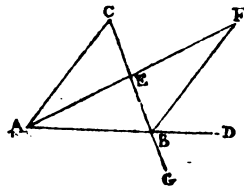
And, if the angle CEB , which is common, be taken away, the remaining angle AEC will be equal to the remaining angle BED (*Ac. 3.*)

And, in the same manner, it may be shewn that the angle AED is equal to the angle CEB . Q. E. D.

COROLL. All the angles made by any number of right lines, meeting in a point, are together equal to four right angles.

P R O P. XVI. THEOREM.

If one side of a triangle be produced, the outward angle will be greater than either of the inward opposite angles.



Let ABC be a triangle, having the side AB produced to D ; then will the outward angle CBD be greater than either of the inward opposite angles BAC or ACB .

For, bisect BC in E (*Prop. 10.*), and join AE ; in which, produced, take EF equal to AE (*Prop. 3.*), and join BF .

Then, since AE is equal to EF , EC to EB (*by Const.*) and the angle AEC to the angle BEF (*Prop. 15.*), the angle ACE will, also, be equal to the angle EBF (*Prop. 4.*)

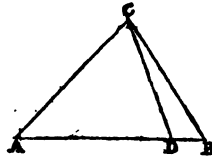
But

But the angle CBD is greater than the angle EBF ; consequently it is also greater than the angle ACE .

And, if CB be produced to G , and AB be bisected, it may be shewn, in like manner, that the angle ABC , or its equal CBD , is greater than CAB . Q. E. D.

PROP. XVII. THEOREM.

The greater side of every triangle is opposite to the greater angle; and the greater angle to the greater side.



Let ABC be a triangle, having the side AB greater than the side AC ; then will the angle ACB be greater than the angle ABC .

For, since AB is greater than AC , let AD be taken equal to AC (*Prop. 3.*), and join CD .

Then, since CDB is a triangle, the outward angle ADC is greater than the inward opposite angle DBC (*Prop. 16.*)

But the angle ACD is equal to the angle ADC , because AC is equal to AD ; consequently the angle ACD is, also, greater than DBC or ABC .

And, since ACD is only a part of ACB , the whole angle ACB must be much greater than the angle ABC .

Again, let the angle ACB be greater than the angle ABC , then will the side AB be greater than the side AC .

For, if AB be not greater than AC , it must be either equal or less.

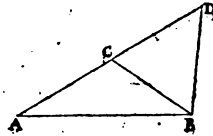
But it cannot be equal, for then the angle ACB would be equal to the angle ABC (*Prop. 5.*), which it is not.

Neither can it be less, for then the angle ACB would be less than the angle ABC (*Prop. 17.*), which it is not.

The side AB , therefore, is neither equal to AC , nor less than it; consequently it must be greater. Q. E. D.

P R O P. XVIII. THEOREM.

Any two sides of a triangle, taken together, are greater than the third side.



Let ABC be a triangle; then will any two sides of it, taken together, be greater than the third side.

For, in AC produced, take CD equal to CB (*Prop. 3.*), and join BD .

Then, because CD is equal to CB (*by const.*), the angle CPB will be equal to the angle CPD (*Prop. 5.*).

But the angle ABD is greater than the angle CPD , consequently it must also be greater than the angle APB .

And, since the greater side of every triangle, is opposite to the greater angle (*Prop. 17.*), the side AD is greater than the side AB .

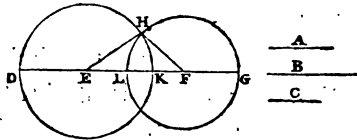
But AD is equal to AC and CD taken, together (*by const.*); therefore AC , CB are also greater than AB .

And,

And, in the same manner, it may be shewn, that any other two sides, taken together, are greater than the third side. Q, E, D.

P R O P, XIX. PROBLEM,

To describe a triangle, whose sides shall be equal to three given right lines, provided any two of them, taken together, be greater than the third.



Let A, B, C be the three given right lines, any two of which, taken together, are greater than the third; it is required to make a triangle whose sides shall be equal to A, B, C respectively.

Draw any right line DG; on which take DE equal to A, EF equal to B, and FG equal to C (*Prop. 3.*)

From the point E, with the distance ED, describe the circle KHD, cutting DG in K; and from the point F, with the distance FG, describe the circle GHL, cutting DG in L.

Then, because EG is greater than ED (*by Hyp.*), or its equal EK, the point G, which is in the circumference of the circle GHL, will fall without the circle KHD.

And, because FD is greater than FG (*by Hyp.*), or its equal FL, the point D, which is in the circumference of the circle KHD, will fall without the circle GHL.

But

But since a part of the circle GHL falls without the circle KHD , and a part of the circle KHD falls without the circle GHL , neither of the circles can be included within the other.

Again, because DE , FG , or their equals EK , FL are, together, greater than EF (*by Hyp.*), the two circles can neither touch nor fall without each other.

They must, therefore, cut one another, in some point H ; and if the right lines EH , FH be drawn, EHF will be the triangle required.

For, since E is the centre of the circle KHD , EH is equal to ED (*Def. 13.*); but ED is equal to A (*by Const.*); therefore EH is also equal to A .

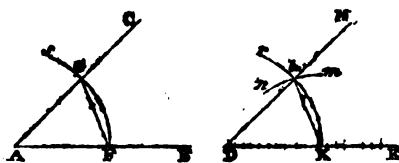
And, because F is the centre of the circle GHL , FH is equal to FG (*Def. 13.*); but FG is equal to C (*by Const.*); therefore FH is also equal to C .

And since EF is, likewise, equal to B (*by Const.*), the three sides of the triangle EHF are respectively equal to the three given lines A , B , C , which was to be shewn.

PROP.

PROP. XX. PROBLEM.

At a given point, in a given right line, to make a rectilinear angle equal to a given rectilinear angle.



Let DE be the given right line, D the given point, and BAC the given rectilinear angle; it is required to make an angle at the point D that shall be equal to BAC .

Take any point F in AB , and from the point A , at the distance AF , describe the circle FG , cutting AC in G ; and join FG .

Make DK equal to AF , and KE equal to FG (*Prop. 3.*); and from the points D, K , at the distances DK, KE , describe the circles KL and KL , cutting each other in L .

Through the points D, L draw the right line DN , and the angle EDN will be equal to BAC , as was required.

For, join KL : then since AG is equal to AF (*Def. 13.*), and AF is equal to DK (*by Const.*), AG will also be equal to DK (*Ax. 1.*).

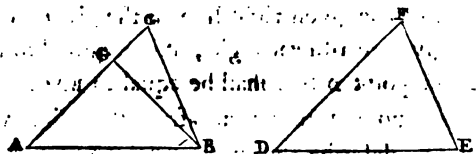
But DK is equal to DL (*Def. 13.*); consequently AG is also equal to DL (*Ax. 1.*); and FG is equal to KE or KL (*by Const.*)

The three sides of the triangle DKL are, therefore, equal to the three sides of the triangle AFG , each to each; whence

whence the angle KDL is equal to the angle FAC , or BAC (*Prop. 7.*); and it is made at the point D , as was to be done.

P R O P. XXI. THEOREM.

If two triangles be mutually equiangular, and have two corresponding sides equal to each other, the other corresponding sides will also be equal, and the two triangles will be equal in all respects.



Let the triangles ABC , DEF be mutually equiangular, and have the side AB equal to the side DE ; then will the side AC be also equal to the side DF , the side BC to the side EF , and the two triangles will be equal in all respects.

For, if AC be not equal to DF , one of them must be greater than the other; let AC be the greater, and make AG equal to DF (*Prop. 3.*); and join BG .

Then, since the two sides AB , AG , are equal to the two sides DE , DF , each to each, and the angle GAB is equal to the angle FDE (*by Hyp.*), the angle ABG will, also, be equal to the angle DEF (*Prop. 4.*)

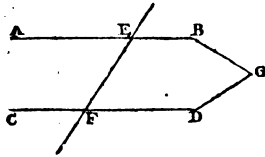
But the angle DEF is equal to the angle ABC (*by Hyp.*); consequently the angle ABG will, also, be equal to the angle ABC , the less to the greater, which is absurd.

The side AC , therefore, cannot be greater than the side DE ; and, in the same manner, it may be shewn that it cannot be less; consequently it must be equal to it.

And, since the two sides AC , AB , are equal to the two sides DE , DF , each to each, and the angle CAB is equal to the angle FDE , the side BC will also be equal to the side EF , and the two triangles will be equal in all respects (*Prop. 4.*) Q. E. D.

PROP. XXII. THEOREM.

If a right line intersect two other right lines, and make the alternate angles equal to each other, those lines will be parallel.



Let the right line EF intersect the two right lines AB , CD , and make the alternate angles AEF , EFD equal to each other; then will AB be parallel to CD .

For, if they be not parallel, let them be produced, and they will meet each other, either on the side AC , or on the side BD (*Def. 20.*)

Suppose them to meet in the point G , on the side BD . Then, since FGE is a triangle, the outward angle AEF is greater than the inward opposite angle EFD (*Prop. 16.*)

But

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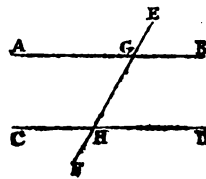
But the angles, $\angle AEF$, $\angle EFB$, are equal to each other (*by Hyp.*); whence they are equal and unequal at the same time, which is absurd.

The lines AB , CD , therefore, cannot meet on the side EB ; and, in the same manner, it may be shewn that they cannot meet on the side AC ; consequently they must be parallel to each other (*Def. 20.*) Q. E. D.

COROLL. Right lines which are perpendicular to the same right line are parallel to each other.

P R O P. XXIII. THEOREM.

If a right line intersect two other right lines, and make the outward angle equal to the inward opposite one, on the same side; or the two inward angles, on the same side, together equal to two right angles, those lines will be parallel.



Let the right line EF intersect the two right lines AB , CD , and make the outward angle $\angle EGB$ equal to the inward angle $\angle CHD$; or the two inward angles $\angle BGH$, $\angle GHD$ together equal to two right angles; then will AB be parallel to CD ,

For,

For, since the angles EGD , GHD are equal to each other (*by Hyp.*), and the angles AGH , EGB are also equal to each other (*Prop. 15.*), the angle AGH will be equal to the angle GHD (*Ax. 1.*).

But when a right line intersects two other right lines, and makes the alternate angles equal to each other, those lines will be parallel (*Prop. 22.*); therefore AB is parallel to CD .

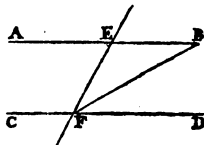
Again, since the angles BGH , GHD are, together, equal to two right angles (*by Hyp.*), and AGH , BGH are, also, equal to two right angles, (*Prop. 13.*) the angles AGH , BGH will be equal to the angles BGH , GHD (*Ax. 1.*)

And, if the common angle BGH be taken away, the remaining angle AGH will be equal to the remaining angle GHD . (*Ax. 3.*)

But these are alternate angles: therefore, in this case, AB will, also, be parallel to CD (*Prop. 22.*) *Q. E. D.*

P R O P. XXIV. THEOREM.

If a right line intersect two parallel right lines it will make the alternate angles equal to each other.



Let the right line EF intersect the two parallel right lines AB , CD ; then will the angle AEF be equal to the alternate angle EFD .

5

For

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For if they be not equal, one of them must be greater than the other; let EFD be the greater; and make the angle EFB equal to AEF (*Prop. 20.*)

Then, since AB , CD are parallel, the right line FB , which intersects CD , being produced, will meet AB in some point B (*Def. 4.*)

And, since EFB is a triangle, the outward angle AEF will be greater than the inward opposite angle EFB (*Prop. 16.*)

But the angles AEF , EFB are equal to each other (*by Const.*) whence they are equal and unequal, at the same time, which is absurd.

The angle EFD , therefore, is not greater than the angle AEF ; and, in the same manner, it may be shewn that it is not less; consequently they must be equal to each other.

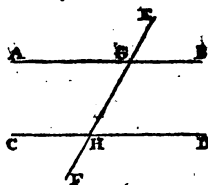
Q. E. D.

COROLL. Right lines which are perpendicular to one of two parallel right lines, are also perpendicular to the other.

PROP.

P R O P. XXV. THEOREM.

If a right line intersect two parallel right lines, the outward angle will be equal to the inward opposite one, on the same side; and the two inward angles, on the same side, will be equal to two right angles.



Let the right line EF intersect the two parallel right lines AB , CD ; then will the outward angle EGB be equal to the inward opposite angle GHD ; and the two inward angles BGH , GHD will be equal to two right angles.

For, since the right line EF intersects the two parallel right lines AB , CD , the angle AGH will be equal to the alternate angle GHD (*Prop. 24.*)

But the angle AGH is equal to the opposite angle EGB (*Prop. 15.*); therefore the angle EGB will, also, be equal to the angle GHD .

Again, since the right line BC falls upon the right line EF , the angles EGB , BGH , taken together, are equal to two right angles (*Prop. 13.*)

But the angle EGB has been shewn to be equal to the angle GHD ; therefore, the angles BGH , GHD , taken together, will, also, be equal to two right angles. Q.E.D.

D

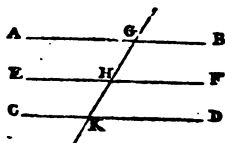
COROLL.

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COROLL. If a right line intersect two other right lines, and make the two inward angles, on the same side, together less than two right angles, those lines, being produced, will meet each other.

P R O P. XXVI. THEOREM.

Right lines which are parallel to the same right line, are parallel to each other.



Let the right lines AB , CD be each of them parallel to EF , then will AB be parallel to CD .

For, draw any right line GK , cutting the lines AB , EF , CD , in the points G , H and K .

Then, because AB is parallel to EF (*by Hyp.*), and GK intersects them, the angle AGH is equal to the alternate angle GHE (*Prop. 24.*)

And because CD is parallel to EF (*by Hyp.*), and GK intersects them, the outward angle GHE is equal to the inward angle HKD (*Prop. 25.*)

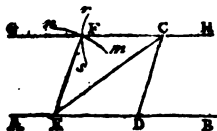
But the angle AGH has been shewn to be equal to the angle GHE ; therefore the angle AGK is also equal to the angle GKD .

And, since the right line GK intersects the two right lines AB , CD , and makes the angle AGK equal to the alternate angle GKD , AB will be parallel to CD , as was to be shewn.

P R O P.

P R O P. XXVII. PROBLEM.

Through a given point, to draw a right line parallel to a given right line.



Let AB be the given right line, and c the given point; it is required to draw a right line through the point c that shall be parallel to AB .

Take any point D in AB , and make DE equal to DC (*Prop. 3.*); and from the points C, E , with the distances CD, ED , describe the arcs rs, nm .

Then, since any two sides of the triangle ECD are, together, greater than the third side (*Prop. 18.*), those arcs will intersect each other (*Prop. 19.*)

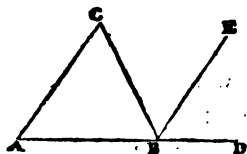
Let them intersect at F ; and through the points F, C , draw the line GH , and it will be parallel to AB , as was required.

For, since the sides CF, FE of the triangle EFC are each equal to the side CD , or DE , of the triangle CDE , (*by Conf.*) and EC is common, the angle ECF will be equal to the angle CED (*Prop. 7.*)

But these are alternate angles; therefore GH is parallel to AB (*Prop. 24.*); and it is drawn through the point c , as was to be done.

P R O P. XXVIII. THEOREM.

If one side of a triangle be produced, the outward angle will be equal to the two inward opposite angles, taken together; and the three angles of every triangle, taken together, are equal to two right angles.



Let ABC be a triangle, having one of its sides AB produced to D ; then will the outward angle CBD be equal to the two inward opposite angles BCA , CAB , taken together; and the three angles BCA , CAB and ABC , taken together, are equal to two right angles.

For through the point B , draw the right line BE parallel to AC (*Prop. 28.*)

Then, because BE is parallel to AC , and CB intersects them, the angle CBE will be equal to the alternate angle BCA (*Prop. 24.*)

And because BE is parallel to AC , and AD intersects them, the outward angle EBD will be equal to the inward angle CAB (*Prop. 25.*)

But the angles CBE , EBD are equal to the whole angle CBD ; therefore the outward angle CBD is equal to the two inward opposite angles BCA , CAB taken together.

And

And if, to these equals, there be added the angle $\angle ABC$, the angles $\angle CBD$, $\angle ABC$, taken together, will be equal to the three angles $\angle BCA$, $\angle CAB$ and $\angle ABC$, taken together.

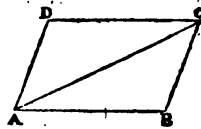
But the angles $\angle CBD$, $\angle ABC$, taken together, are equal to two right angles (*Prop. 13.*); consequently the three angles $\angle BCA$, $\angle CAB$ and $\angle ABC$, taken together, are also equal to two right angles.

COROLL. 1. If two angles of one triangle, be equal to two angles of another, each to each, the remaining angles will also be equal.

COROLL. 2. Any quadrilateral may be divided into two triangles; therefore all the four angles of such a figure, taken together, are equal to four right angles.

P R O P. XXIX. THEOREM.

Right lines joining the corresponding extremes of two equal and parallel right lines are themselves equal and parallel.



Let AB , DC be two equal and parallel right lines; then will the right lines AD , BC , which join the corresponding extremes of those lines, be also equal and parallel.

For draw the diagonal, or right line AC :

Then, because AB is parallel to DC , and AC intersects them, the angle $\angle DCA$ will be equal to the alternate angle $\angle CAB$ (*Prop. 24.*)

D 3

And,

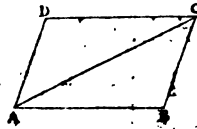
And, because AB is equal to DC (*by Hyp.*), AC common to each of the triangles ABC , ADC , and the angle DCA equal to the angle CAB , the side AD will also be equal to the side BC , and the angle DAC to the angle ACB (*Prop. 4.*)

Since, therefore, the right line AC intersects the two right lines AD , BC , and makes the alternate angles equal to each other, those lines will be parallel (*Prop. 23.*)

But the line AD has been proved to be equal to the line BC ; consequently they are both equal and parallel. *Q.E.D.*

P R O P. XXX. THEOREM.

The opposite sides and angles of any parallelogram are equal to each other, and the diagonal divides it into two equal parts.



Let $ABCD$ be a parallelogram, whose diagonal is AC ; then will its opposite sides and angles be equal to each other, and the diagonal AC will divide it into two equal parts.

For, since the side AD is parallel to the side BC (*Def. 22.*), and the right line AC intersects them, the angle DAC will be equal to the alternate angle ACB (*Prop. 24.*)

And, because the side DC is parallel to the side AB (*Def. 22.*), and AC intersects them, the angle DCA will be equal to the alternate angle CAB (*Prop. 24.*)

Since, therefore, the two angles DAC , DCA , are equal to the two angles ACB , CAB , each to each, the remaining angle

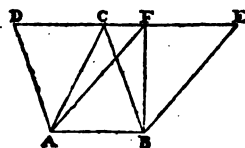
angle ADC will also be equal to the remaining angle ABC (*Prop. 29. Cor.*) and the whole angle DAB to the whole angle DCB .

But, the triangles CDA , ABC , being mutually equiangular, and having AC common, the side DC will also be equal to the side AB , and the side AD to the side BC , and the two triangles will be equal in all respects (*Prop. 21.*)

Q. E. D.

PROP. XXXI. THEOREM.

Parallelograms, and triangles, standing upon the same base, and between the same parallels, are equal to each other.



Let AE , BD be two parallelograms standing upon the same base AB , and between the same parallels AB , DE ; then will the parallelogram AE be equal to the parallelogram BD .

For, since AD is parallel to BC (*Def. 22.*), and DE intersects them, the outward angle ECB will be equal to the inward opposite angle FDA (*Prop. 25.*)

And, because AF is parallel to BE (*Def. 22.*), and DE intersects them, the outward angle AFD will be equal to the inward opposite angle BEC (*Prop. 25.*)

Since, therefore, the angle ECB is equal to the angle FDA , and the angle AFD to the angle BEC , the remaining

angle

D

angle

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angle CBE will be equal to the remaining angle DAF (*Prop. 29. Cor. 1.*)

But the side AD is also equal to the side BC (*Prop. 31.*); consequently, since the triangles ADF, BCE are mutually equiangular, and have two corresponding sides equal to each other, they will be equal in all respects (*Prop. 21.*)

If, therefore, from the whole figure ABED, there be taken the triangle BCE, there will remain the parallelogram BD; and if, from the same figure, there be taken the triangle ADF, there will remain the parallelogram AE.

But if equal things be taken from the same thing, the remainders will be equal; consequently, the parallelogram AE is equal to the parallelogram BD.

Again, let ABC, ABF be two triangles, standing upon the same base AB, and between the same parallels, AB, CF; then will the triangle ABC be equal to the triangle ABF.

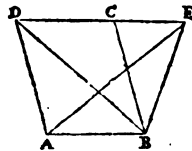
For produce CF, both ways, to D and E, and draw AD parallel to BC, and BE to AF (*Prop. 28.*)

Then, since BD, AE, are two parallelograms, standing upon the same base AB, and between the same parallels AB, DE, they are equal to each other (*Prop. 32.*)

And, because the diagonals AC, BF bisect them (*Prop. 31.*), the triangle ABC will also be equal to the triangle ABF. Q.E.D.

P R O P. XXXII. THEOREM.

If a parallelogram and a triangle stand upon the same base, and between the same parallels, the parallelogram will be double the triangle,



Let the parallelogram AC and the triangle AEB stand upon the same base AB, and between the same parallels AB, DE; then will the parallelogram AC be double the triangle AEB.

For join the points B, D; then will the parallelogram AC be double the triangle ADB, because the diagonal DB divides it into two equal parts (*Prop. 31.*)

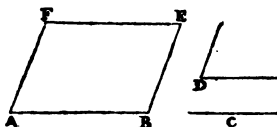
But the triangle ADB is equal to the triangle AEB, because they stand upon the same base AB, and between the same parallels AB, DE (*Prop. 32.*); whence the parallelogram AC is also double the triangle AEB. Q.E.D.

COROLL. If the base of the parallelogram be half that of the triangle, or the base of the triangle be double that of the parallelogram, the two figures will be equal to each other.

P R O P.

P R O P. XXXIII. PROBLEM.

To make a parallelogram that shall have its opposite sides equal to two given right lines, and one of its angles equal to a given rectilineal angle.



Let AB and c be two given right lines, and D a given rectilineal angle; it is required to make a parallelogram that shall have its opposite sides equal to AB and c , and one of its angles equal to D .

At the point A , in the line AB , make the angle BAF equal to the angle D (*Prop. 20.*) and the side AF equal to c (*Prop. 3.*)

Also, make FE parallel and equal to AB (*Prop. 28 and 3.*), and join BE ; then will AE be the parallelogram required.

For, since FE is parallel and equal to AB (*by Const.*), BE will be parallel and equal to AF (*Prop. 30.*); whence the figure AE is a parallelogram.

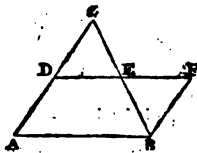
And, because AF is equal to c (*by Const.*) BE will also be equal to c ; and the angle BAF was made equal to the angle D .

The opposite sides of the parallelogram AE are, therefore, equal to the two given lines AB and c ; and one of its angles is equal to the given angle D , as was to be done.

P R O P.

PROP. XXXIV. THEOREM.

If two sides of a triangle be bisected, the right line joining the points of bisection, will be parallel to the base, and equal to one half of it,



Let ABC be a triangle, whose sides CA , CB are bisected in the points D , E ; then will the right line DE , joining those points, be parallel to AB , and equal to one half of it.

For, in DE produced, take EF equal to ED (*Prop. 3.*), and join BF :

Then, since EC is equal to EB (*by Hyp.*) ED to EF (*by Const.*) and the angle DEC to the angle BEF (*Prop. 15.*), the side BF will also be equal to the side DC , or its equal DA , and the angle EFB to the angle EDC , (*Prop. 4.*)

And, because the right line DF intersects the two right lines CD , FB , and makes the angle EDC equal to the alternate angle EFB , BF will be parallel to DC or DA (*Prop. 24.*)

The right lines BF , AD , therefore, being equal and parallel, the lines DE , AB , joining their extremes, will also be equal and parallel. (*Prop. 30.*)

But DF is the double of DE (*by Const.*); consequently AB is also the double of DE ; that is DE is the half of AB .

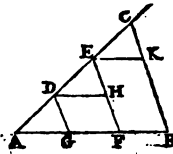
Q. E. D.

PROP.

44. ELEMENTS OF GEOMETRY.

P R O P. XXXV. . PROBLEM.

To divide a given finite right line into any proposed number of equal parts.



Let AB be the given right line; it is required to divide it into a certain proposed number of equal parts.

From the point A, draw any right line AC, in which take the equal parts AD, DE, EC, at pleasure, (*Prop. 3.*) to the number proposed.

Join BC; and parallel thereto draw the right lines EF, DG, (*Prop. 28.*) cutting AB, in F and G; then will AB be divided into the same number of equal parts with AC, as was required.

For take EH, CK, each equal to DG (*Prop. 3.*), and join D, H and E, K.

Then, since DG is parallel to EF (*by Conf.*), and AE intersects them, the outward angle ADG will be equal to the inward opposite angle DEH (*Prop. 25.*)

And, because the sides AD, DG of the triangle AGD, are equal to the sides DE, EH of the triangle DHE (*by Conf.*), and the angle ADG is equal to the angle DEH, the base AG will also be equal to the base DH, and the angle DAG to the angle EDH (*Prop. 4.*)

But, since the right line AE intersects the two right lines, AG, DH, and makes the outward angle EDH equal

to the inward opposite angle DAG , DH will be parallel to AG or GF (*Prop. 23.*)

And, in the same manner it may be shown, that EK is equal to AG , and parallel to AG or FB .

The figures GH , FK , therefore, being parallelograms, the side DH will be equal to the side GF , and the side EK to the side FB (*Prop. 31.*)

But DH , EK have been each proved to be equal to AG ; consequently GF , FB are, also, each equal to AG ; whence the line AB is divided into the same number of equal parts with AC , as was to be done.

BOOK II.

DEFINITIONS.

1. A rectangle is a parallelogram whose angles are all right angles.



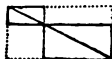
2. A square is a rectangle, whose sides are all equal to each other.



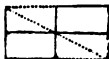
3. Every rectangle is said to be contained by any two of the right lines which contain one of the right angles.



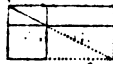
4. If two right lines be drawn through any point in the diagonal of a parallelogram, parallel to its opposite sides, the figures which are intersected by the diagonal are called parallelograms about the diagonal.



5. And the other two parallelograms, which are not intersected by the diagonal, are called complements to the parallelograms which are about the diagonal.



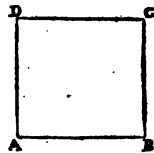
6. In every parallelogram, either of the two parallelograms about the diagonal, together with the two complements, is called a gnomon.



7. The altitude of any figure is a perpendicular drawn from the vertical angle to the base.

PROP. I PROBLEM.

UPON a given right line to describe a square.



Let AB be the given right line ; it is required to describe a square upon it.

Make AD, BC, each perpendicular and equal to AB (I. 11 and 3.), and join DC ; then will AC be the square required.

For, since the angles DAB, ABC are right angles (*by Const.*), AD will be parallel to BC (I. 22 *Cor.*)

And because AD, BC are equal and parallel, AB, DC will, also, be equal and parallel (I. 30.)

But AD, BC are each equal to AB (*by Const.*) ; whence AD, AB, BC and CD are all equal to each other.

The

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The figure AC, therefore, is an equilateral parallelogram; and it has, likewise, all its angles right angles.

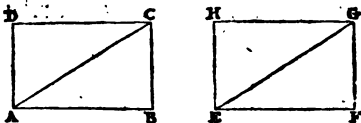
For the angle DAB is equal to the angle DCB, and the angle ABC to the angle ADC (I. 30.)

But the angles DAB, ABC are right angles (*by Constr.*); consequently the angles DCB, ADC are, also, right angles.

The figure AC, therefore, being both equilateral and rectangular, is a square; and it is described upon the line AB, as was to be done.

P R O P. II. THEOREM.

Rectangles and Squares contained under equal lines are equal to each other.



Let BD, FH be two rectangles, having the sides AB, BC equal to the sides EF, FG, each to each; then will the rectangle BD be equal to the rectangle FH:

For draw the diagonals AC, EG:

Then, since the two sides AB, BC are equal to the two sides EF, FG, each to each (*by Hyp.*), and the angle B is equal to the angle F (I. 8.), the triangle ABC will be equal to the triangle EFG (I. 4.)

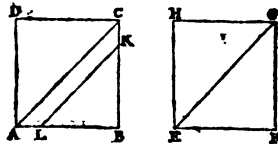
But the diagonal of every parallelogram divides it into two equal parts (I. 30.); whence the halves being equal; the wholes will also be equal:

The rectangle BD is, therefore, equal to the rectangle FH; and in the same manner it may be proved when the figures are squares. Q.E.D.

P R O P.

P R O P. III. THEOREM.

The sides and diagonals of equal squares are equal to each other.



Let BD, FH, be two equal squares; then will the side AB be equal to the side EF, and the diagonal AC to the diagonal EG.

For if AB, EF be not equal, one of them must be greater than the other; let AB be the greater, and make BL, BK each equal to EF or FG (I. 3.); and join LK.

Then, because BL is equal to FE, BK to FG, and the angle LBK to the angle EFG, being each of them right angles, the triangle BLK will be equal to the triangle FEG (I. 4.)

But the triangle FEG is equal to the triangle BAC, being each of them the halves of the equal squares FH, BD (I. 30.); whence the triangle BLK is also equal to the triangle BAC, the less to the greater, which is absurd.

The side AB, therefore is not greater than the side EF; and in the same manner it may be proved that it cannot be less; consequently they are equal to each other.

And because AB is equal to EF, BC to FG, and the angle ABC to the angle EFG (I. 8.), the side AC will also be equal to the side EG (I. 4.)

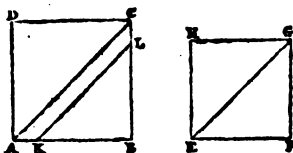
Q. E. D.

E

PROP.

P R O P. IV. THEOREM.

The square of a greater line is greater than the square of a less; and the greater square has the greater side.



Let the right line AB be greater than the right line EF; then will BD, the square of AB, be greater than FH, the square of EF.

For since AB is greater than EF, and BC than FG (*by Hyp.*), take BK, a part of BA, equal to EF, and BL, a part of BC, equal to FG (I. 3.); and join KL.

Then, because BK is equal to FE, BL to FG, and the angle KBL to the angle EFG (I. 8.), the triangle BLK will be equal to the triangle FGE (I. 4.)

But the triangle BCA is greater than the triangle BLK, whence it is also greater than the triangle FGE.

And since the square BD is double the triangle BCA, and the square FH is double the triangle FGE (I. 30.), the square BD will also be greater than the square FH.

Again, let the square BD be greater than the square FH; then will the side AB be greater than the side EF.

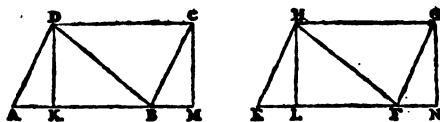
For if AB be not greater than EF, it must be either equal to it, or less; but it cannot be equal to it, for then the square

square BD would be equal to the square FH (II. 2.), which it is not.

Neither can it be less, for then the square BD would be less than the square FH (II. 4.), which it is not; consequently AB is greater than EF , as was to be shewn.

P R O P. V. THEOREM.

Parallelograms or triangles, having equal bases and altitudes, are equal to each other.



Let AC , EG be two parallelograms, having the base AB equal to the base EF , and the altitude DK to the altitude HL ; then will the parallelogram AC be equal to the parallelogram EG .

For upon AB , EF , produced if necessary, let fall the perpendiculars CM , GN (I. 12.)

Then, since MD , NH are rectangular parallelograms, the side DC is equal to the side KM , and the side HG to the side LN (I. 30.)

But DC is also equal to AB , and HG to EF (I. 30.); therefore KM is equal to AB , and LN to EF .

And, since AB is equal to EF (*by Hyp.*), KM will be equal to LN ; and consequently the rectangle MD is equal to the rectangle NH (II. 2.)

But the rectangle MD is equal to the parallelogram AC , because they stand upon the same base DC , and between the same parallels DC , AM .

E 2

And,

And, for the same reason, the rectangle NH is equal to the parallelogram EG ; whence the parallelogram AC is equal to the parallelogram EG .

Again, let ABD , EFH be two triangles, having the base AB equal to the base EF , and the altitude DK to the altitude HL ; then will the triangle ABD be equal to the triangle EFH .

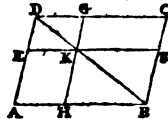
For, if the parallelograms AC , EG be completed, they will be equal to each other; by the former part of the proposition.

And since the diagonals, DB , HF divide them into two equal parts (I. 30.) the triangle ABD will also be equal to the triangle EFH . Q. E. D.

COROLLARY. Parallelograms or Triangles standing upon equal bases, and between the same parallels, are equal to each other.

P R O P. VI. THEOREM.

The complements of the parallelograms which are about the diagonal of any parallelogram are equal to each other.



Let AC be a parallelogram, and AK , KC , complements about the diagonal BD ; then will the complement AK be equal to the complement KC .

For,

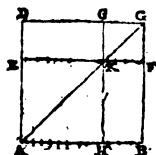
For since AC is a parallelogram, whose diagonal is BD , the triangle DAB will be equal to the triangle BCD (I. 30.)

And, because EG , HF are also parallelograms, whose diagonals are DK , KB , the triangle DGK will be equal to the triangle DEK , and the triangle KFB to the triangle KHB (I. 30.)

But, since the triangles DGK , KFB are together, equal to the triangles DEK , KHB , and the whole triangle DAB to the whole triangle DCB , the remaining part AK will be equal to the remaining part KC . Q. E. D.

P R O P. VII. THEOREM.

Parallelograms which are about the diagonal of a square are themselves squares.



Let BD be a square, and HE , FG parallelograms about its diagonal AC ; then will those parallelograms also be squares.

For since the side of the square AB is equal to the side BC , the angle CAB will be equal to the angle ACB (I. 5.)

And because the right line GH is parallel to the right line CB , the angle AKH will also be equal to the angle ACB (I. 25.)

The angles CAB , AKH are, therefore, equal to each other; and consequently the side AH is equal to the side HK (I. 6.)

E 3

But

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But the side AH is equal to the side EK , and the side HK to the side AE (I. 30.); whence the figure HE is equilateral.

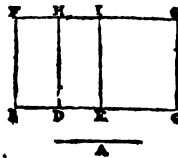
It has also all its angles right angles :

For EAH is a right angle, being the angle of a square ; and HC , EF are each of them parallel to the sides of the same square, whence the remaining angles will also be right angles (I. 25.)

The figure HE , therefore, being equilateral, and having all its angles right angles, is a square ; and the same may be proved of the figure FG , Q. E. D.

P R O P. VIII. THEOREM.

The rectangles contained under a given line and the several parts of another line, any how divided, are, together, equal to the rectangle of the two whole lines.



Let A and BC be two right lines, one of which, BC , is divided into several parts in the points D , E ; then will the rectangle of A and BC , be equal to the sum of the rectangles of A and BD , A , and DE , and A and EC .

For make BF perpendicular to BC (I. 11.) and equal to A (I. 3.), and draw FG parallel to BC , and DH , EI and

and CG each parallel to BF (I. 27.) producing them till they meet FG in the points H, I, G .

Then, since the rectangle BH is contained by BD and BF (II. Def. 3.), it is also contained by BD and A , because BF is equal to A (*by Const.*)

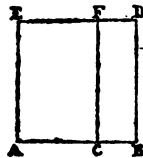
And, since the rectangle DI is contained by DE and DH , it is also contained by DE and A , because DH is equal to BF (I. 30.), or A .

The rectangle EG , in like manner, is contained by EC and A ; and the rectangle BG by BC and A .

But the whole rectangle BC , is equal to the rectangles BH, DI and EG , taken together; whence the rectangle of A and BC is also equal to the rectangles of A and BD, A and DE and A and EC , taken together. Q. E. D.

P R O P. IX. THEOREM.

If a right line be divided into any two parts, the rectangles of the whole line and each of the parts, are, together, equal to the square of the whole line.



Let the right line AB be divided into any two parts in the point C ; then will the rectangle of AB, AC , together with the rectangle of AB, BC , be equal to the square of AB .

E 4

For,

For, upon AB describe the square AD (II. 1.), and through C draw CF parallel to AE or BD (I. 27.)

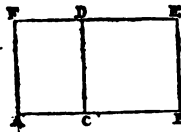
Then, since the rectangle AF is contained by AE , AC , it is also contained by AB , AC , because AE is equal to AB (II. Def. 2.)

And, since the rectangle CD is contained by BD , BC , it is also contained by AB , BC , because BD is equal to AB .

But AD , or the square of AB , is equal to the rectangles AF , CD , taken together; whence the rectangle of AB , AC , together with the rectangle of AB , BC , is also equal to the square of AB . Q. E. D.

P R O P. X. THEOREM.

If a right line be divided into any two parts, the rectangle of the whole line and one of the parts, is equal to the rectangle of the two parts, together with the square of the aforefaid part.



Let the right line AB be divided into any two parts in the point C ; then will the rectangle of AB , BC be equal to the rectangle of AC , CB , together with the square of CB .

For upon CB describe the square CE (II. 1.), and through A draw AF parallel to CD (I. 27.) meeting ED , produced, in F .

Then,

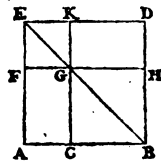
Then, since AE is a rectangle, contained by AB , BE , it is also contained by AB , BC , because BE is equal to BC (II. Def. 2.)

And, in like manner, AD is a rectangle contained by AC , CD , or by AC , CB ; and CE is the square of CB (by Const.)

But the rectangle AE is equal to the rectangle AD , and the square CE , taken together; whence the rectangle of AB , BC is also equal to the rectangle of AC , CB together with the square of CB . Q. E. D.

PROP. XI. THEOREM.

If a right line be divided into any two parts, the square of the whole line will be equal to the squares of the two parts, together with twice the rectangle of those parts.



Let the right line AB be divided into any two parts in the point C ; then will the square of AB be equal to the squares of AC , CB together with twice the rectangle of AC , CB .

For upon AB make the square AD (II. 1.), and draw the diagonal EB ; and make CK , FH parallel to AE , ED (I. 27.)

Then,

Then, since the parallelograms about the diagonal of a square are themselves squares (II. 7.), FK will be the square of FG , or its equal AC , and CH of CB .

And since the complements of the parallelograms about the diagonal are equal to each other (II. 6.), the complement AG will be equal to the complement GD .

But AG is equal to the rectangle of AC , CB , because CG is equal to CB (II. Def. 2.); and GD is also equal to the rectangle of AC , CB , because GK is equal to GF (Def. II. 2.) or AC (I. 30.), and CH to CB (I. 30.).

The two rectangles AG , GD are, therefore, equal to twice the rectangle of AC , CB ; and FK , CH have been proved to be equal to the squares of AC , CB .

But these two rectangles, together with the two squares, make up the whole square AD ; consequently the square AD is equal to the squares of AC , CB , together with twice the rectangle of AC , CB .

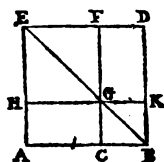
Q. E. D.

COROLL. If a line be divided into two equal parts, the square of the whole line will be equal to four times the square of half the line.

PROP.

P R O P. XII. THEOREM.

If a right line be divided into any two parts, the squares of the whole line, and one of the parts, are equal to twice the rectangle of the whole line and that part, together with the square of the other part.



Let the right line AB be divided into any two parts in the point C ; then will the squares of AB , BC , be equal to twice the rectangle of AB , BC together with the square of AC .

For, upon AB make the square AD (II. 1.), and draw the diagonal BE ; and make FC , HK parallel to BD , BA (I. 27.):

Then because AG is equal to GD (II. 6.), to each of these equals add CK , and the whole AK will be equal to the whole CD .

And, since the doubles of equals are equal, the gnomon HBF , together with CK , will be the double of AK .

But CK is a square upon CB (II. 7.), and twice the rectangle of AB , BC is the double of AK , whence the gnomon HBF , together with the square CK , is, also, equal to twice the rectangle AB , BC .

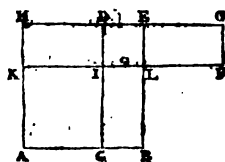
And,

And, because HF is a square upon HG or AC (II. 7.); if this be added to each of these equals, the gnomon HBF , together with the squares CK , HF , will be equal to twice the rectangle AB , BC , together with the square of AC .

But the gnomon HBF , together with the squares CK , HF , are equal to the whole square AD , together with the square CK ; consequently, the squares of AB , BC , are equal to twice the rectangle of AB , BC together with the square of AC . Q. E. D.

P R O P. XIII. THEOREM.

The difference of the squares of any two unequal lines, is equal to a rectangle under their sum and difference.



Let AB , AC be any two unequal lines; then will the difference of the squares of those lines be equal to a rectangle under their sum and difference.

For upon AB , AC make the squares AE , AI (II. 1.); and in HE , produced, take EG equal to AC (I. 3.); and make GF parallel to EB (I. 27.); and produce CI , IK till they meet HG , GF in D and F .

Then, since HE is equal to AB (Def. II. 2.) and EG to AC (by Const.), HG will be equal to the sum of AB and AC .

And

And because AH is equal to AB , and AK to AC (II. Def. 2.), KH will be equal to CB , or the difference of AB and AC .

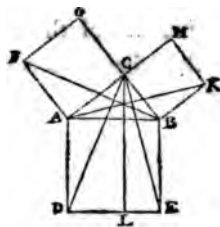
But the rectangle KC is contained by HG and HK ; whence it is, also, contained by the sum and difference of AB and AC .

And, since LE is equal to HK (I. 30.) or CB (*by Const.*), and EG to AC (*by Const.*) CI , or LB , the rectangle IG will be equal to the rectangle LC (II. 2.)

But the rectangles HL , LC are, together, equal to the difference of the squares AE , AI ; consequently the rectangles HL , LG , or the whole rectangle KC , is also equal to the difference of those squares. Q. E. D.

PROP. XIV. THEOREM.

In any right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.



Let ABC be a right angled triangle, having the right angle ACB ; then will the square of the hypotenuse AB be equal to the sum of the squares of AC and CB .

For, on AB , describe the square $AEDB$ (II. 1.), and on AC , CB the squares ACF , BHK ; and through the point C , draw

draw CL parallel to AD or BE (I. 27.) and join BF , CD , AK and CE .

Then, since the right line AC meets the two right lines GC , OB in the point C , and makes each of the angles ACG , ACB a right angle (*by Hyp. and Def. 2.*), GC will be in the same right line with CB (I. 14.)

And, because the angle FAC is equal to the angle DAB (I. 8.), if the angle CAB be added to each of them, the whole angle FAB will be equal to the whole angle DAC .

The sides FA , AB , are, also, equal to the sides CA , AD , each to each, (*Def. 2.*), and their included angles have, likewise, been shewn to be equal; whence the triangle ABF is equal to the triangle ACD (I. 4.)

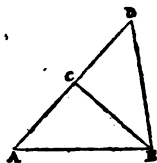
But the square AG is double the triangle ABF (I. 32.) and the parallelogram AL is double the triangle ACD (I. 32.); consequently the parallelogram AL is equal to the square AG (*Ax. 6.*)

And, in the same manner, it may be demonstrated, that the parallelogram BL is equal to the square BH ; therefore the whole square AB is equal to the squares AG and BH taken together. Q. E. D.

COROLL. The difference of the squares of the hypotenuse and either of the other sides is equal to the square of the remaining side.

[P R O P. XV. THEOREM.

If the square of one of the sides of a triangle be equal to the sum of the squares of the other two sides, the angle contained by those sides will be a right angle.



Let ABC be a triangle; then if the square of the side AB be equal to the sum of the squares of AC , CB , the angle ACB will be a right angle.

For, at the point C , make CD at right angles to CB (I. 11.), and equal to AC (I. 3.); and join DB .

Then, since the squares of equal lines are equal (II. 2.), the square of DC will be equal to the square of AC .

And, if, to each of these equals, there be added the square of CB , the squares of DC , CB will be equal to the squares of AC , CB .

But the squares of DC , CB are equal to the square of DB (II. 14.), and the squares of AC , CB to the square of AB (*by Hyp.*); whence the square of DB is equal to the square of AB .

And since equal squares have equal sides (II. 3.), AB is equal to DB ; BC is also common to each of the triangles ABC , DBC , and AC is equal to CD (*by Constr.*);

con-

62. ELEMENTS OF GEOMETRY.

draw CL parallel to AD or BE (I. 27.) and join BF , CD , AK and CE .

Then, since the right line AC meets the two right lines GC , OB in the point C , and makes each of the angles ACG , ACB a right angle (*by Hyp. and Def. 2.*), GC will be in the same right line with CB (I. 14.)

And, because the angle FAC is equal to the angle DAB (I. 8.), if the angle CAB be added to each of them, the whole angle FAB will be equal to the whole angle DAC .

The sides FA , AB , are, also, equal to the sides CA , AD , each to each, (*Def. 2.*), and their included angles have, likewise, been shewn to be equal; whence the triangle ABF is equal to the triangle ACD (I. 4.)

But the square AG is double the triangle ABF (I. 32.) and the parallelogram AL is double the triangle ACD (I. 32.); consequently the parallelogram AL is equal to the square AG (*Ax. 6.*)

And, in the same manner, it may be demonstrated, that the parallelogram BL is equal to the square BH ; therefore the whole square AB is equal to the squares AG and BH taken together. Q. E. D.

COROLL. The difference of the squares of the hypotenuse and either of the other sides is equal to the square of the remaining side.

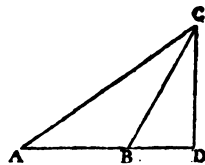
PROP.

But things which are equal to the same thing are equal to each other ; consequently the difference of the squares of AC, CB is equal to the difference of the squares of AD, DB. Q.E.D.

COROLL. The rectangle under the sum and difference of the two sides of any triangle, is equal to the rectangle under the base and the difference of the segments of the base (II. 13.)

P R O P. XVII. THEOREM.

In any obtuse-angled triangle, the square of the side subtending the obtuse angle, is greater than the sum of the squares of the other two sides, by twice the rectangle of the base and the distance of the perpendicular from the obtuse angle.



Let ABC be a triangle, of which $\angle A$ is an obtuse angle, and CD perpendicular to AB ; then will the square of AC be greater than the squares of AB, BC, by twice the rectangle of AB, BD.

For, since the right line AD is divided into two parts, in the point B, the square of AD is equal to the squares of

AB,

BD,

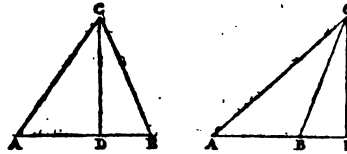
64 ELEMENTS OF GEOMETRY.

consequently the angle ACB is equal to the angle BCD (I. 7.)

But the angle BCD is a right angle (*by Const.*), whence the angle ACB is also a right angle. Q. E. D.

PROP. XVI. THEOREM.

The difference of the squares of the two sides of any triangle, is equal to the difference of the squares of the two lines, or distances, included between the extremes of the base and the perpendicular.



Let ABC be a triangle, having CD perpendicular to AB ; then will the difference of the squares of AC , CB be equal to the difference of the squares of AD , DB .

For the sum of the squares of AD , DC is equal to the square of AC (II. 14.); and the sum of the squares of BD , DC is equal to the square of BC (II. 14.)

The difference, therefore, between the sum of the squares of AD , DC , and the sum of the squares of BD , DC is equal to the difference of the squares of AC , CB .

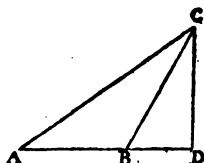
And, since DC is common, the difference between the sum of the squares of AD , DC , and the sum of the squares of BD , DC is equal to the difference of the squares of AD , DB .

But things which are equal to the same thing are equal to each other ; consequently the difference of the squares of AC, CB is equal to the difference of the squares of AD, DB. Q.E.D.

COROLL. The rectangle under the sum and difference of the two sides of any triangle, is equal to the rectangle under the base and the difference of the segments of the base (II. 13.)

P R O P. XVII. THEOREM.

In any obtuse-angled triangle, the square of the side subtending the obtuse angle, is greater than the sum of the squares of the other two sides, by twice the rectangle of the base and the distance of the perpendicular from the obtuse angle.



Let ABC be a triangle, of which $\angle A$ is an obtuse angle, and CD perpendicular to AB ; then will the square of AC be greater than the squares of AB, BC, by twice the rectangle of AB, BD.

For, since the right line AD is divided into two parts, in the point B, the square of AD is equal to the squares of

AB,

BD,

Then, because $\angle AEC$ is an obtuse angle, the square of AC is equal to the squares of AE , EC together with twice the rectangle of AE , ED (II. 17.)

And, because $\angle BEC$ is an acute angle, the square of CB together with twice the rectangle of BE , ED is equal to the squares of BE , EC (II. 18.)

And since AE is equal to EB (*by Const.*), the square of BC together with twice the rectangle of AE , ED is equal to the squares of AE , EC .

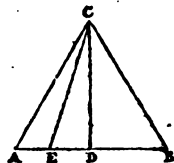
But if equals be added to equals, the wholes will be equal; whence the squares of AC , CB , together with twice the rectangle of AE , ED , are equal to twice the squares of AE , EC , together with twice the rectangle of AE , ED .

And, if twice the rectangle of AE , EC , which is common, be taken away, the sum of the squares of AC , CB will be equal to twice the sum of the squares of AE , EC .

Q. E. D.

PROP. XX. THEOREM.

In an isosceles triangle, the square of a line drawn from the vertex to any point in the base, together with the rectangle of the segments of the base, is equal to the square of one of the equal sides of the triangle.



Let ABC be an isosceles triangle, and CE a line drawn from the vertex to any point in the base AB ; then will the square

square of CE , together with the rectangle of AE , EB be equal to the square of AC or CB .

For bisect the base AB in D (I. 10.), and join the points C , D .

Then, since AC is equal to CB , AD to DB , and CD is common to each of the triangles ACD , BCD , the angle CDA will be equal to the angle CDB (I. 7.); and consequently CD will be perpendicular to AB (*Def.* 8, 9.)

And, because ACE is a triangle, and CD is the perpendicular, the difference of the squares of AC , CE is equal to the difference of the squares of AD , DE (II. 16.)

But, since BE is the sum of AD and DE , and AE is their difference, the difference of the squares of AD , DE is equal to the rectangle of AE , EB ; consequently, the difference of the squares of AC , CE is also equal to the rectangle AE , EB .

And if, to each of these equals, there be added the square of CE , the square of AC will be equal to the square of CE together with the rectangle of AE , EB .

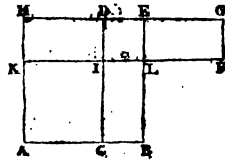
Q. E. D.

And, because HF is a square upon HG or AC (II. 7.); if this be added to each of these equals, the gnomon HBF, together with the squares CK, HF, will be equal to twice the rectangle AB, BC, together with the square of AC.

But the gnomon HBG , together with the squares CK , HF , are equal to the whole square AD , together with the square CK ; consequently, the squares of AB , BC , are equal to twice the rectangle of AB , BC together with the square of AC . Q. E. D.

PROP. XIII. THEOREM.

The difference of the squares of any two unequal lines, is equal to a rectangle under their sum and difference.



Let AB, AC be any two unequal lines; then will the difference of the squares of those lines be equal to a rectangle under their sum and difference.

For upon AB, AC make the squares AE, AI (II. 1.); and in HE, produced, take HE equal to AC (I. 3.); and make GF parallel to EB (I. 27.); and produce CI, IK till they meet HG, GF in D and F.

Then, since HE is equal to AB (*Def. II. 2.*) and EG to AC (*by Conf.*), HG will be equal to the sum of AB and AC .

And

And because AH is equal to AB , and AK to AC (II. Def. 2.), KH will be equal to CB , or the difference of AB and AC .

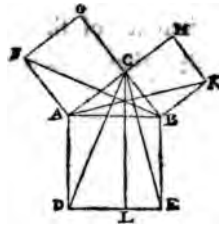
But the rectangle KC is contained by HG and HK ; whence it is, also, contained by the sum and difference of AB and AC .

And, since LE is equal to HK (I. 30.) or CB (by Const.), and EG to AC (by Const.) CI , or LB , the rectangle IG will be equal to the rectangle LC (II. 2.)

But the rectangles HL , LC are, together, equal to the difference of the squares AE , AI ; consequently the rectangles HL , LC , or the whole rectangle KC , is also equal to the difference of those squares. Q. E. D.

P R O P. XIV. THEOREM.

In any right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

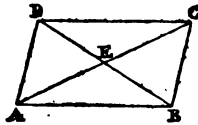


Let ABC be a right angled triangle, having the right angle ACB ; then will the square of the hypotenuse AB be equal to the sum of the squares of AC and CB .

For, on AB , describe the square AE (II. 1.), and on AC , CB the squares AC , BH ; and through the point C , draw

PROP. XXI. THEOREM.

The diagonals of any parallelogram bisect each other, and the sum of their squares is equal to the sum of the squares of the four sides of the parallelogram.



Let $ABCD$ be a parallelogram, whose diagonals AC , BD intersect each other in E ; then will AE be equal to EC , and BE to ED ; and the sum of the squares of AC , BD will be equal to the sum of the squares of AB , BC , CD and DA .

For since AB , DC are parallel, and AC , BD intersect them, the angle DCE will be equal to the angle EAB (I. 24.), and the angle CDE to the angle EBA (I. 24.)

The angle DEC is likewise equal to the angle AEB (I. 15.), and the side DC to the side AB (I. 30.); consequently DE is also equal to EB , and CE to EA (I. 21.)

Again, since DB is bisected in E , the sum of the squares of DC , CB will be equal to twice the sum of the squares of DE , EC (II. 19.)

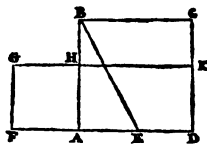
And, because DC is equal to AB , and CB to DA (I. 30.) the sum of the squares of AB , CB , DC and DA are equal to four times the sum of the squares of DE , EC .

But four times the square of DE is equal to the square of BD (II. 11. Cor.), and four times the square of EC is equal

equal to the square of AC; whence the sum of the squares of AC, BD are equal to the sum of the squares of AB, BC, CD and DA.
Q.E.D.

P R O P. XXII. PROBLEM.

To divide a given right line into two parts, so that the rectangle contained by the whole line and one of the parts, shall be equal to the square of the other part.



Let AB be the given right line; it is required to divide it into two parts, so that the rectangle of the whole line and one of the parts shall be equal to the square of the other part.

Upon AB describe the square AC (II. 1.), and bisect the side of it AD in E (I. 10.)

Join the points B, E; and, in EA produced, take EF equal to EB (I. 3.); and upon AF describe the square FH (II. 1.)

Then will AB be divided in H so, that the rectangle of AB, BH, will be equal to the square of AH.

For, since DF is equal to the sum of EB and ED, or its equal EA, and AF is equal to their difference, the rectangle of DF, FA is equal to the difference of the squares of EB, EA (II. 13.)

F 4

But

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But the rectangle of DF , FA is equal to DC , because FA is equal to FC (II. *Def.* 2.); and the difference of the squares of EB , EA is equal to the square of AB (II. 14. *Cor.*); whence DC is equal to AC .

And, if from each of these equals, the part DH , which is common to both, be taken away, the remainder AG will be equal to the remainder HC .

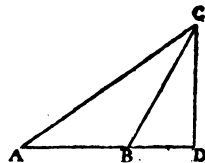
But HC is the rectangle of AB , BH ; for AB is equal to BC ; and AG is the square of AH ; therefore the right line AB is divided in H so, that the rectangle of AB , BH is equal to the square of AH , which was to be done.

But things which are equal to the same thing are equal to each other ; consequently the difference of the squares of AC, CB is equal to the difference of the squares of AD, DB. Q. E. D.

COROLL. The rectangle under the sum and difference of the two sides of any triangle, is equal to the rectangle under the base and the difference of the segments of the base (II. 13.)

P R O P. XVII. THEOREM.

In any obtuse-angled triangle, the square of the side subtending the obtuse angle, is greater than the sum of the squares of the other two sides, by twice the rectangle of the base and the distance of the perpendicular from the obtuse angle.



Let ABC be a triangle, of which ABC is an obtuse angle, and CD perpendicular to AB ; then will the square of AC be greater than the squares of AB, BC, by twice the rectangle of AB, BD.

For, since the right line AD is divided into two parts, in the point B, the square of AD is equal to the squares of

F

AB,

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6. A segment of a circle, is a figure contained under any arc and the chord of that arc.



7. A tangent to a circle, is a right line which passes through a point in the circumference without cutting it.



8. Right lines, or chords, are said to be equally distant from the centre of a circle, when perpendiculars drawn to them from the centre are equal.



9. And the right line on which the greater perpendicular falls, is said to be farther from the centre,



10. An angle in a segment, is that which is contained by two right lines, drawn from any point in the arc of the segment, to the two extremities of the chord of that arc.



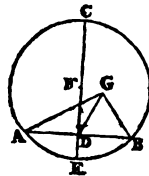
11. One circle is said to touch another, when it passes through a point in its circumference without cutting it.



P R O P.

PROP. I. PROBLEM.

To find the centre of a given circle.



Let ABC be the given circle ; it is required to find its centre.

Draw any chord AB, and bisect it in D (I. 10.) ; and through the point D draw CE at right angles to AB (I. 11.), and bisect it in F : then will the point F be the centre of the circle.

For if it be not, some other point must be the centre, either in the line EC, or out of it.

But it cannot be any other point in the line EC, for if it were, two lines drawn from the centre of the circle to its circumference would be unequal, which is absurd.

Neither can it be any point out of that line ; for if it can, let G be that point ; and join GA, GD and GB.

Then, because GA is equal to GB (I. Def. 13.), AD to DB (by Const.), and GD common to each of the triangles AGD, BGD, the angle ADG will be equal to the angle BDG (I. 7.)

But when one line falls upon another, and makes the adjacent angles equal, those angles are, each of them, right angles (I. Def. 8 and 9.)

The

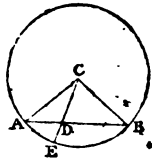
The angle ADG , therefore, is equal to the angle ADC (I. 8.), the whole to the part, which is absurd; consequently no point but F can be the centre of the circle.

Q. E. D.

COROLL. If any chord of a circle be bisected, a right line drawn through that point, perpendicular to the chord, will pass through the centre of the circle.

P R O P. II. THEOREM.

If any two points be taken in the circumference of a circle, the chord, or right line which joins them, will fall wholly within the circle.



Let ABE be a circle, and A, B any two points in the circumference; then will the right line AB , which joins those points, fall wholly within the circle.

For find C , the centre of the circle ABE (III. 1.), and join C, A, C, B ; and through any point D , in AB , draw the right line CE , cutting the circumference in E .

Then, because CA is equal to CB (I. Def. 13.), the angle CAB will be equal to the angle CBA (I. 5.)

And, since the outward angle CDB of the triangle ACD , is greater than the inward opposite angle CAB (I. 16.), it will also be greater than the angle CBA .

But

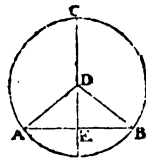
But the greater side of every triangle is opposite to the greater angle (I. 17.); whence CB , or its equal CE , will be greater than CD .

The point D , therefore, falls within the circle; and the same may be shewn of any other point in AB ; consequently the whole line AB must fall within the circle.

Q.E.D.

P R O P. III. THEOREM.

If a right line, which passes through the centre of a circle, bisect a chord, it will be perpendicular to it; and if it be perpendicular to the chord, it will bisect it.



Let ABC be a circle, and CE a right line which passes through the centre D , and bisects the chord AB in E ; then will CE be perpendicular to AB .

For join the points AD , DB :

Then, because AD is equal to DB (II. Def. 13.), AE to EB (by Hyp.), and ED common to each of the triangles ADE , BDE , the angle DEA will be equal to the angle DEB (I. 7.)

But one line is said to be perpendicular to another, when it makes the angles on both sides of it equal to each other (I. Def. 8.); consequently CE is perpendicular to the chord AB .

Again,

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Again, let the right line DE be drawn from the centre D , perpendicular to the chord AB ; then will AB be bisected in the point E .

For join the points AD , DB , as before :

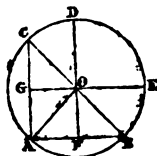
Then, since the angle DAB is equal to the angle DBA (I. 5.), and the angle AED to the angle DEB , (being each of them right angles) the angle ADE will also be equal to the angle EDB (I. 28. *Cor.* 1.)

And, because the triangles DEA , DEB are mutually equiangular, and have the side DE common, the side AE will also be equal to the side EB (I. 21.); whence AB is bisected in the point E , as was to be shewn.

COROLL. If a right line be drawn from the vertex of an isosceles triangle, to the middle of the base, it will be perpendicular to it; and if it be perpendicular to the base, it will bisect both it and the vertical angle.

P R O P. IV. THEOREM.

If more than two equal right lines can be drawn from any point in a circle to the circumference, that point will be the centre.



x Let $ABDC$ be a circle, and O point within it; then if any three right lines OA , OB , OC , drawn from the point O to the circumference, be equal to each other, that point will be the centre.

For

For draw the lines AB , AC , and bisect them in the points F , E (I. 10.); and through the centre O , draw FD , GE , cutting the circumference in D and E .

Then, since AF is equal to FB (*by Const.*), AO to OB (*by Hyp.*), and OF common to each of the triangles AOF , BOF , the angle AFO will be equal to the angle BFO (I. 7.)

And because the right line OF falls upon the right line AB , and makes the adjacent angles equal to each other, DF will be perpendicular to AB (I. Def. 8.)

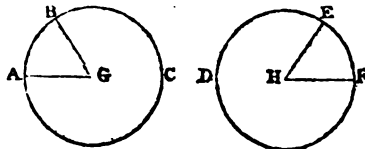
But when a right line bisects any chord at right angles, it passes through the centre of the circle (III. 1. Cor.); whence the centre must be somewhere in the line FD .

And, in the same manner, it may be shewn, that the centre must be somewhere in the line GE .

But the lines FD , GE have no other point but O which is common to them both; therefore O is the centre of the circle ABD , as was to be shewn.

P R O P. V. THEOREM.

Circles of equal radii are equal to each other; and if the circles are equal, the radii will be equal.



Let ABC , DEF be two circles, of which the radii GA , GB are equal to the radii HF , HE ; then will the circle ABC be equal to the circle DEF .

For

For conceive the circle DEF. to be applied to the circle ABC, so that the centre H may coincide with the centre G.

Then, since the radii HF, HE are equal to the radii GA, GB (*by Hyp.*), the points F, E will fall in the circumference of the circle ABC (*1. Def. 13.*); and the same may be shewn of any other point D.

But since any number of points, taken in the circumference of the circle DEF, fall in the circumference of the circle ABC, the two circumferences must coincide, and consequently the circles are equal to each other.

Again, let the circle ABC be equal to the circle DEF; then will the radii GA, GB be equal to the radii HF, HE.

For if they be not equal, they must be either greater or less: let them be greater; and apply the circles to each other as before.

Then, since the radii GA, GB are greater than the radii HF, HE, the points F, E will fall within the circle ABC; and the same may be shewn of any other point D.

But, since any number of points, taken in the circumference of the circle DEF, fall within the circle ABC, the whole circle DEF must, also, fall within the circle ABC.

The circle DEF is, therefore, less than the circle ABC, and equal to it at the same time (*by Hyp.*), which is absurd: whence the radii GA, GB are not greater than the radii HF, HE.

And in the same manner it may be shewn that they cannot be less; consequently they are equal to each other.

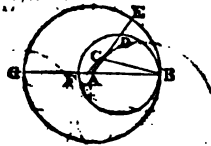
Q. E. D.

COROLL. Equal circles, or such as have equal radii, or diameters, have equal circumferences.

P R O P.

PROP. VI. THEOREM.

If two circles touch each other internally, the centres of the circles and the point of contact will be all in the same right line.



Let the two circles BEC, BDF touch each other internally at the point B; then will the centres of those circles and the point B be in the same right line.

For let A be the centre of the circle BEC, and draw the diameter CB.

And if the centre of the circle BDF be not in CB, let, if possible, some point C, out of that line, be the centre; and join A, C, C, B; and produce AC to cut the circles in D and E.

Then, since ACB is a triangle, the sides AC, CB, taken together, are greater than the side AB (I. 18.), or its equal AE.

And if, from these equals, the part AC, which is common, be taken away, the remainder CB will be greater than the remainder CE.

But, since C is the centre of the circle BDF (by Hyp.), CB is equal to CD (I. Def. 13.); whence CD will also be greater than CE, which is impossible.

The point C, therefore, cannot be the centre of the circle BDF; and the same may be shewn of any other point out of the line AB.

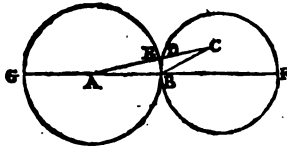
Q. E. D.

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P R O P. VII. THEOREM.

If two circles touch each other externally, the centres of the circles and the point of contact will be all in the same right line.



Let the two circles BEG, BDF touch each other externally at the point B; then will the centres of those circles and the point B, be in the same right line.

For, let A be the centre of the circle BEG, and draw the diameter GB, which produce till it cuts the circle BDF in F.

And, if the centre of the circle BDF be not in the line AF, let, if possible, some point C, out of that line, be the centre; and join C, A, C, B.

Then, since A is the centre of the circle BEG, AE is equal to AB (I. Def. 13.)

And because C is the centre of the circle BDF (*by Hyp.*) CD is equal to CB (I. Def. 13.)

But AB, BC, together, are greater than AC (I. 18.); therefore AE, CD, together, are also greater than AC; which is absurd.

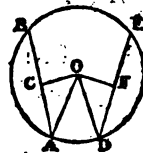
The point C, therefore, cannot be the centre of the circle BDF; and the same may be shewn of any other point out of the line AF.

Q. E. D.

P R O P.

P R O P. VIII. THEOREM.

Any two chords in a circle, which are equally distant from the centre, are equal to each other; and if they be equal to each other, they will be equally distant from the centre.



Let $ABED$ be a circle, whose centre is O ; then will any two chords AB , DE , which are equally distant from O , be equal to each other.

For join the points AO , OD , and let fall the perpendiculars OC , OF (I. 12.)

Then, since a right line, drawn from the centre of a circle, at right angles to any chord, bisects it (III. 3.), AC will be equal to CB , and DF to FE .

And, because the angles ACO , DFO are right angles, the squares of AC , CO will be equal to the square of AO (II. 14.), and the squares of DF , FO to the square of OD .

But the square of AO is equal to the square of OD (II. 2.); consequently the squares of AC , CO will be equal to the squares of DF , FO .

And since OC is equal to OF (III. Def. 8.), the square of OC will be equal to the square of OF (II. 2.); whence

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the remaining square of AC will also be equal to the remaining square of DF ; or AC equal to DF (II. 3.), and AB to DE (I. Ax. 6.)

Again, let the chord AB be equal to the chord DE ; then will OC , OF , or their distances from the centre, be equal to each other.

For the squares of AC , CO are equal to the square of OA (II. 14.), and the squares of DF , FO to the square of OD .

But the square of OA is equal to the square of OD (II. 2.); therefore the squares of AC , CO are equal to the squares of DF , FO .

And since AC is the half of AB (III. 3.), and DF is the half of DE (III. 3.), the square of AC is equal to the square of DF (II. 2.)

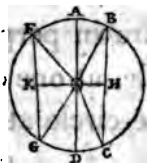
The remaining square of CO is, therefore, equal to the remaining square of FO ; and consequently CO is equal to FO (II. 3.), as was to be shewn.

COROLL. If two right angled triangles, having equal hypotenuses, have two other sides also equal, the remaining sides will likewise be equal, and the triangles will be equal in all respects.

PROP.

PROP. IX. THEOREM.

A diameter is the greatest right line that can be drawn in a circle, and, of the rest, that which is nearer the centre is greater than that which is more remote.



Let ABCD be a circle, of which the diameter is AD, and the centre O; then if BC be nearer the centre than FG, AD will be greater than BC, and BC than FG.

For draw OH, OK perpendicular to BC, FG (I. 12.), and join OB, OC, OF and OG.

Then, because OA is equal to OB (I. Def. 13.), and OD to OC, AD is equal to OB and OC taken together.

But OB, OC, taken together, are greater than BC (I. 18.); therefore AD is also greater than BC.

Again, the squares of OH, HB are equal to the square of OB (II. 14.), and the squares of OK, KF to the square of OF.

But the square of OB is equal to the square of OF (II. 2.); whence the squares of OH, HB are equal to the squares of OK, KF.

And since FG is farther from the centre than BC (by Hyp.), OK will be greater than OH (III. Def. 9), and the square of OK than the square of OH (II. 4.)

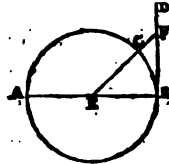
The remaining square of HB , therefore, is greater than the remaining square of KF , and HB greater than KF (II. 4.)

But BC is the double of BH , and FG is the double of FK (III. 3.); consequently BC is also greater than FG .

Q. E. D.

PROP. X. THEOREM.

A right line drawn perpendicular to the diameter of a circle, at one of its extremities, is a tangent to the circle at that point.



Let ABC be a circle whose centre is E , and diameter AB ; then if DB be drawn perpendicular to AB , it will be a tangent to the circle at the point B .

For in BD take any point F , and draw EF , cutting the circumference of the circle in C .

Then, since the angle EBD is a right angle (*by Hyp.*), the angles BEF , EFB , will be each of them less than a right angle (I. 28.)

And, because the greater side of every triangle is opposite to the greater angle (I. 17.), the side EF is greater than the side EB , or its equal EC .

But since EF is greater than EC , the point F will fall without the circle ABC ; and the same may be shewn of any other point in BD , except B ,

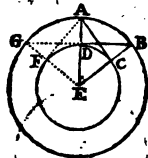
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The line BD , therefore, cannot cut the circle, but must fall wholly without it, and be a tangent to it at the point B , as was to be shewn.

SCHOLIUM. A right line cannot touch a circle in more than one point, for if it met it in two points it would fall wholly within the circle (III. 2.)

PROP. XI. THEOREM.

From a given point to draw a tangent to a given circle.



Let A be the given point, and FDC the given circle; it is required from the point A to draw a tangent to the circle FDC .

Find E, the centre of the circle FDC (III. 1.), and join EA; and from the point E, at the distance EA, describe the circle GAB.

Through the point D, draw DB at right angles to EA (I. 11.); and join EB, AC; and AC will be the tangent required.

For, since E is the centre of the circles FDC, GAB, EA is equal to EB, and ED to EC.

And, because the two sides EA , EC , of the triangle EAC , are equal to the two sides EB , ED , of the triangle EBD , and the angle E is common, the angle ECA will also be equal to the angle EDB (I. 4.)

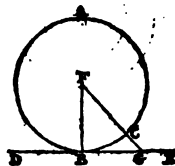
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But the angle $\angle EDB$ being a right angle, the angle $\angle ECA$ is also a right angle; therefore since AC is perpendicular to the diameter EC , it will touch the circle EDC , and be a tangent to it at the point C (III. 10.)

Q. E. I.

PROP. XII. THEOREM.

If a right line be a tangent to a circle, and another right line be drawn from the centre to the point of contact, it will be perpendicular to the tangent.



Let the right line DE be a tangent to the circle ABC at the point B , and, from the centre F , draw the right line FB ; then will FB be perpendicular to DE .

For if it be not, let, if possible, some other right line FG be perpendicular to DE .

Then, because the angle $\angle FGB$ is a right angle (*by Hyp.*), the angle $\angle FBG$ will be less than a right angle (I. 28.)

And, since the greater side of every triangle is opposite to the greater angle (I. 17.), the side FB will be greater than the side FG .

But FB is equal to FC ; therefore FC will also be greater than FG , a part greater than the whole, which is impossible.

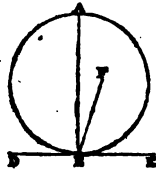
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The line FC , therefore, cannot be perpendicular to DE ; and the same may be demonstrated of any other line but FB ; consequently FB is perpendicular to DE .

Q. E. D.

PROP. XIII. THEOREM.

If a right line be a tangent to a circle, and another right line be drawn at right angles to it, from the point of contact, it will pass through the centre of the circle.



Let the right line DE be a tangent to the circle ACB at the point B ; then if AB be drawn at right angles to DE , from the point of contact B , it will pass through the centre of the circle.

For if it does not, let F , if possible, be the centre of the circle; and join FB .

Then, since DE is a tangent to the circle, and FB is a right line drawn from the centre to the point of contact, the angle FBE is a right angle (III. 12.)

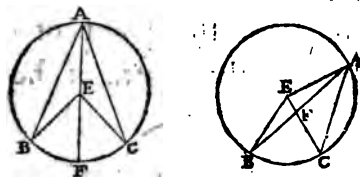
But the angle ABE is also a right angle, by construction; whence the angle FBE is equal to the angle ABE ; the less to the greater, which is impossible.

The point F , therefore, is not the centre; and the same may be shewn of any other point which is out of the line AB ; consequently AB must pass through the centre of the circle, as was to be shewn.

PROP.

P R O P. XIV. THEOREM.

An angle at the centre of a circle is double to that at the circumference, when both of them stand upon the same arc.



Let the angle BEC be an angle at the centre of the circle ABC , and BAC an angle at the circumference, both standing upon the same arc BC ; then will the angle BEC be double the angle BAC .

First, let E , the centre of the circle, be within the angle BAC ; and draw AE , which produce to F .

Then, because EA is equal to EB , the angle EAB will be equal to the angle EBA (I. 5.)

And, because AEB is a triangle, the outward angle BEF will be equal to the two inward opposite angles EAB , EBA , taken together (I. 28.)

But since the angles EAB , EBA , are equal to each other, they are, together, double the angle EAB ; whence the angle BEF is also double the angle EAB .

And, in the same manner it may be shewn, that the angle FEC is double the angle EAC ; consequently the whole angle BEC will also be double the whole angle BAC .

Again,

Again, let E , the centre of the circle ABC , fall without the angle BAC , and join AE .

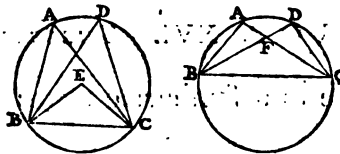
Then, since the angle BFE , of the triangle EFB , is equal to the angle CFA of the triangle CAF (I. 15.) the remaining angles BEF , FBE of the one, are, together, equal to the remaining angles FAC , FCA of the other (I. 28.)

But the angle FBE is equal to the angle EAF (I. 5.); and the angle FCA , or ECA , to the angle EAC (I. 5.); therefore the angles BEF , EAF , are, together, equal to the angles FAC , EAC .

And, if the angle EAF , which is common, be taken away, the remaining angle BEF or BEC , will be equal to twice the angle FAC , or BAC , as was to be shewn.

P R O P. XV. THEOREM.

All angles in the same segment of a circle, or which stand on the same arc, are equal to each other.



Let $ABCD$ be a circle, and BAC , BDC any two angles in the same segment $BADC$; then will the angles BAC , BDC be equal to each other.

For, first, let the segment $BADC$ be greater than a semicircle, and having found the centre E , join BE and EC ,

Then,

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Then, since an angle at the centre of a circle is double to that at the circumference (III. 14.), the angle BAC will be half the angle BEC .

And, for the same reason, the angle BDC will, also, be half the angle BEC .

But things which are halves of the same thing are equal to each other; consequently the angle BAC is equal to the angle BDC .

Again, let the segment $BADC$ be not greater than a semicircle:

Then, since the right lines BD , AC intersect each other in F , the angle BFA will be equal to the opposite angle DFC (I. 15.)

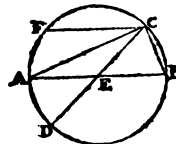
And because the segment $ABCD$ is greater than a semicircle, the angles ABD , ACD , which stand in that segment, are equal to each other (III. 15.)

But since the two angles BFA , ABF of the triangle FBA , are equal to the two angles DFC , FCD of the triangle DCF , the remaining angle BAF , or BAC , will also be equal to the remaining angle FDC , or BDC .

Q. E. D.

P R O P. XVI. THEOREM.

An angle in a semicircle is a right angle.



Let ABC be a semicircle; then will any angle ACB , in that semicircle, be a right angle.

For,

For, find the centre of the circle E (III. 1.) and draw the diameter CE .

Then, because an angle at the centre of a circle is double to that at the circumference (III. 14.) the angle AED will be double the angle ACD .

And, for the same reason, the angle BED will be double the angle BCD .

The angles AED , BED , therefore, taken together, are double the whole angle ACB .

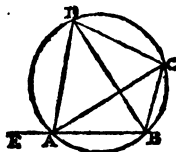
But the angles AED , BED , are, together, equal to two right angles (I. 13.); consequently the angle ACB will be equal to one right angle. Q. E. D.

COROLL. The angle BAC , which stands in a segment greater than a semi-circle, is less than a right angle (I. 28.):

And the angle BCF , which stands in a segment less than a semi-circle, is greater than a right angle.

PROP. XVII. THEOREM.

The opposite angles of any quadrilateral figure, inscribed in a circle, are equal to two right angles.



Let $ABCD$ be a quadrilateral, inscribed in the circle $AECB$; then will the opposite angles BAD , BCD , taken together, be equal to two right angles.

For,

For, draw the diagonals AC , BD , and produce the side BA to E .

Then, because the outward angle of any triangle, is equal to the two inward opposite angles taken together (I. 28.), the angle EAD will be equal to the angles ABD , ADB .

And, because all angles in the same segment of a circle are equal to each other (III. 15.), the angle ABD will be equal to the angle ACD , and the angle ADB to the angle ACB .

The angle EAD , therefore, which is equal to the angles ABD , ADB , taken together, will also be equal to the angles ACD , ACB , taken together, or to the whole angle BCD .

But the angles EAD , BAD , taken together, are equal to two right angles (I. 13.); consequently the angles BCD , BAD , taken together, will also be equal to two right angles.

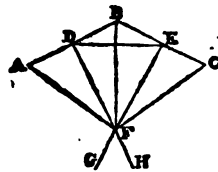
Q. E. D.

COROLL. If any side AB , of the quadrilateral $ABCD$, be produced, the outward angle EAD will be equal to the inward opposite angle BCD .

PROP.

P R O P. XVIII. THEOREM.

Through any three points, not situate in the same right line, to describe the circumference of a circle.



Let A, B, C, be any three points, not situate in the same right line ; it is required to describe the circumference of a circle through those points.

Draw the right lines AB, BC, and bise^ct them with the perpendiculars DH, EG (I. 10 and 11.) ; and join DE.

Then, because the angles FED, FDE are less than two right angles, the lines DH, EG will meet each other, in some point F (Cor. I. 25.) ; and that point will be the centre of the circle required.

For, draw the lines FA, FB and FC,

Then, since AD is equal to DB, DF common, and the angle ADF equal to the angle FDB (I. 8.), the side FA will also be equal to the side FB (I. 4.)

And, in the same manner, it may be shewn, that the side FC is also equal to the side FB.

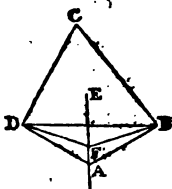
The lines FA, FB and FC, are, therefore, all equal to each other ; and consequently F is the centre of a circle which will pass through the points A, B and C, as was to be shewn.

SCHO.

SCHO. If the segment of a circle be given, and any three points be taken in the circumference, the centre of the circle may be found, as above.

PROPOSITION XIX. THEOREM.

If the opposite angles of a quadrilateral, taken together, be equal to two right angles, a circle may be described about that quadrilateral.



Let $ABCD$ be a quadrilateral, whose opposite angles DCB , DAB are, together, equal to two right angles: then may a circle be described about that quadrilateral.

For since the circumference of a circle may be described through any three points (III. 18.), let E be the centre of a circle which passes through the points B , C , A ; and draw the indefinite right line EFA .

And if the circle does not pass through the fourth point A , let it pass, if possible, through some other point F , in the line EA , and draw the lines DF , FB , and BD .

Then, since the opposite angles BFD , DCB are, together, equal to two right angles (III. 17.), and the angles BAD , DCB are also equal to two right angles (*by Hyp.*), the angles BFD , DCB will be equal to the angles BAD , DCB .

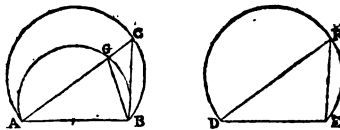
And

And if, from each of these equals, there be taken the angle DCB, which is common to both, the remaining angle BAD will be equal to the remaining angle BFD; or, which is the same thing, the two angles DFE, EFB will be equal to the two angles DAE, EAB, which is impossible (I. 16.)

The circumference of the circle, therefore, cannot pass through the point F; and the same may be demonstrated of any other point in the line EA, except the point A; whence a circle may be described about the quadrilateral ABCD, as was to be shewn.

PROP. XX. THEOREM.

Segments of circles, which stand upon equal chords, and contain equal angles, are equal to each other.



Let ACB, DFE be two segments of circles, which stand upon the equal chords AB, DE, and contain equal angles; then will those segments be equal to each other.

For let the segment DFE be applied to the segment ACB, so that the point D may fall upon the point A, and the line DE upon the line AB.

Then, since DE is equal to AB (*by Hyp.*), the point E will fall upon the point B, and the two segments will coincide with each other.

H

For

For if they do not, there must be some point, in the circumference of one of them, which will fall either within or without the other.

Let the point F , in the circumference of the circle DFE , be that point; which suppose to fall at G within the circle ACB ; and draw the lines AGC , BC and BG .

Then, since the outward angle ACB , of the triangle BCC , is greater than the inward opposite angle CCB , it will also be greater than the angle DFE , which is equal to CCB , or ACB (*by Hyp.*)

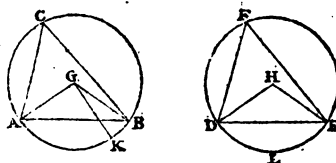
But the angle AGB is also equal to the angle DFE , because the segments in which they stand are identical; whence they are equal and unequal at the same time, which is absurd.

The point F , therefore, cannot fall within the circle ACB ; and in the same manner it may be shewn that it cannot fall without it; consequently the segments must coincide, and be equal to each other. Q. E. D.

COROLL. Segments of circles, which stand upon equal chords, and contain equal angles, have equal circumferences.

P R O P. XXI. THEOREM.

In equal circles, equal angles stand upon equal arcs; whether they be at the centres or circumferences; and if the arcs be equal, the angles will be equal.



Let ABC , DEF be two equal circles, having the angles AGB , DHE , at their centres, equal to each other, as also the angles ACB , DFE ; at their circumferences; then will the arc AKB be equal to the arc DLE .

For, join the points AB , DE ; then, since the circles are equal to each other (*by Hyp.*), their radii and circumferences will also be equal (III. 5.)

And, since the two sides AG , GB of the triangle AGB , are equal to the two sides DH , HE of the triangle DEH , and the angle AGB to the angle DHE (*by Hyp.*); their bases AB , DE will likewise be equal (I. 4.)

The chord AB , therefore, being equal to the chord DE , and the angle ACB to the angle DFE (*by Hyp.*), the arc BCA will also be equal to the arc EFD (*Cor.* III. 20.)

But since the whole circumference of the circle ABC , is equal to the whole circumference of the circle DEF , and the arc BCA to the arc EFD , the arc AKB will also be equal to the arc DLE .

H 2

Again,

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Again, let the arc AKB be equal to the arc DLE ; then will the angle AGB be equal to the angle DHE , and the angle ACB to the angle DFE .

For, if AGB be not equal to DHE , one of them must be greater than the other; let AGB be the greater; and make the angle AGK equal to DHE (I. 20.)

Then, since equal angles stand upon equal arcs (III. 21.), the arc AK will be equal to the arc DLE .

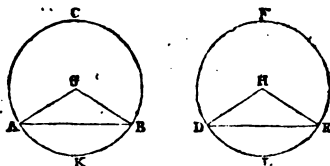
But the arc DLE is equal to the arc AKB (*by Hyp.*); whence the arc AK is also equal to the arc AKB ; the less to the greater, which is impossible.

The angle AGB , therefore, is not greater than the angle DHE ; and in the same manner it may be proved that it cannot be less; consequently they are equal to each other.

And since angles at the centre are double to those at the circumference, the angle ACB will also be equal to the angle DFE .
Q.E.D.

P R O P. XXII. THEOREM.

In equal circles, equal chords subtend equal arcs; the greater equal to the greater, and the less to the less; and if the chords be equal the arcs will be equal.



Let ABC , DEF be two equal circles, in which the chord AB is equal to the chord DE ; then will the arc ACB

ACB be equal to the arc DFE, and the arc AKB to the arc DLE.

For, find G, H, the centres of the circles (III. 1.), and join GA, GB, HD and HE.

Then, since the circles are equal to each other (*by Hyp.*) their radii and circumferences will also be equal (III. 5.)

And, since the sides AG, GB are equal to the sides DH, HE, and the base AB to the base DE (*by Hyp.*), the angle AGB will also be equal to the angle DHE (I. 22.)

But equal angles, at the centres of equal circles, stand upon equal arcs (III. 21.); therefore the arc AKB is equal to the arc DLE.

And since the whole circumference of the circle ABC is equal to the whole circumference of the circle DEF, and the arc AKB to the arc DLE, the arc ACB will also be equal to the arc DFE.

Again, let ABC, DEF be two equal circles, of which the arc AKB is equal to the arc DLE; then will the chord AB be equal to the chord DE.

For let G, H be the centres of the circles, found as before; and join AG, GB, DH and HE.

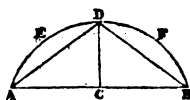
Then, since the circles are equal to each other (*by Hyp.*), the radii AG, GB will be equal to the radii DH, HE (III. 5.)

And because the arc AKB is equal to the arc DLE (*by Hyp.*), the angles AGB, DHE, at the centres, will be equal (III. 21.)

But, since the two sides AG, GB are equal to the two sides DH, HE, and the angle AGB to the angle DHE, the base AB will also be equal to the base DE (I. 4.) Q.E.D.

PROP. XXIII. PROBLEM.

To bisect a given arc, that is, to divide it into two equal parts.



Let ADB be the given arc ; it is required to divide it into two equal parts.

Draw the right line AB , which bisect in c (I. 10.); and, from the point c , erect the perpendicular CD (I. 11.); then will the arc ADB be bisected in the point D , as was required.

For, join the points AD , DB ; then, since the two sides AC , CD , of the triangle ADC , are equal to the two sides BC , CD of the triangle BDC , and the angle ACD to the angle BCD (I. 8.), the base AD will be equal to the base DB (I. 4.).

And, because DC , or DC produced, passes through the centre of the circle (III. 1 *Cor.*), the segments ADF , DBF will be each of them less than a semicircle.

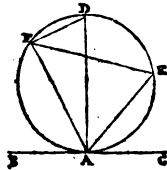
But equal chords are subtended by equal arcs, the greater equal to the greater, and the less to the less (III. 22.); whence the chord AD being equal to the chord DB , the arc AED will be equal to the arc DFB . Q. E. F.

SCHOLIUM. An arc of a circle cannot, in general, be trisected, or divided into three equal parts, by any known method, which is purely geometrical.

PROP.

P R O P. XXIV. THEOREM.

The angle formed by a tangent to a circle and a chord drawn from the point of contact, is equal to the angle in the alternate segment.



Let BC be a tangent to the circle $AFDE$, and AE a chord drawn from the point of contact; then will the angle CAE be equal to the angle AFE in the alternate segment.

For draw the diameter AD (III. 1.) and join the points F, D :

Then, because BC is a tangent to the circle, and AD is a line drawn through the centre, from the point of contact, the angle DAC will be a right angle (III. 12.)

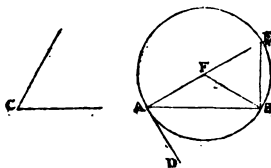
And, because AFD is a semi-circle, the angle DFA will also be a right angle (III. 16.), and equal to the angle DAC .

But since all angles in the same segment of a circle are equal to each other (III. 15.), the angle DFE will be equal to the angle DAE .

If, therefore, from the equal angles DAC, DFA , there be taken the equal angles DAE, DFE , the remaining angle CAE will be equal to the remaining angle AFE . Q.E.D.

P R O P. XXV. PROBLEM.

Upon a given right line to describe a segment of a circle, that shall contain an angle equal to a given rectilineal angle,



Let AB be the given right line, and C the given rectilineal angle ; it is required to describe a segment of a circle upon the line AB that shall contain an angle equal to C .

Make the angle BAD equal to C (I. 20.), and, from the point A , draw AE at right angles to AD (I. 11.) and make the angle ABF equal to the angle FAB (I. 20.)

Then, since the angles ABF , FAB are equal to each other, and less than two right angles, the sides AF , FB will meet, and be equal to each other (I. 25 *Cor.* and I. 6.)

From the point F , therefore, at the distance FA , or FB , describe the circle AEB , and AEB will be the segment required.

For let AF be produced to cut the circle in E ; and join the points E , B .

Then, because AD is perpendicular to the diameter AE , it will be a tangent to the circle at the point A (III. 10.)

And, because AD touches the circle, and AB is a chord drawn from the point of contact, the angle BAD will

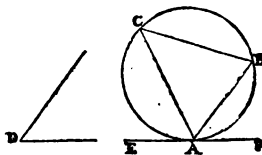
will be equal to the angle AEB in the alternate segment (III. 24.)

But the angle BAD is equal to the angle C, by construction; consequently the angle AEB is also equal to the angle C, Q.E.D.

SCHOLIUM. When the given angle is a right angle, a semi-circle described upon the given line will be the segment required (III. 16.)

PROP. XXVI. PROBLEM.

To cut off a segment from a given circle, that shall contain an angle equal to a given rectilineal angle,



Let ABC be a given circle, and D a given rectilineal angle; it is required to cut off a segment from the circle ABC, that shall contain an angle equal to D.

Draw the right line EF, to touch the circle ABC in the point A (III. 10.), and make the angle FAB equal to the angle D (I. 20.); then will ABCA be the segment required.

For, since EF is a tangent to the circle, and AB is a chord drawn from the point of contact, the angle FAB will be equal to the angle ACB in the alternate segment (III. 24.)

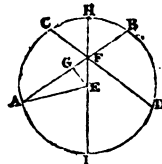
But

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But the angle FAB is equal to the angle D , by construction; consequently the angle ACB , in the segment $ABCA$, is also equal to the angle D . Q. E. I.

P R O P. XXVII. THEOREM.

If two right lines in a circle intersect each other, the rectangle contained under the segments of the one, will be equal to the rectangle contained under the segments of the other.



Let AB , CD be any two right lines, in the circle $ACBD$, intersecting each other in the point F ; then will the rectangle contained under the parts AF , FB of the one, be equal to the rectangle contained under the parts CF , FD of the other.

For, through the point F , draw the diameter HI (III. 1.); and, from the centre E , draw EG at right angles to AB (I. 12.), and join AE :

Then, since AEF is a triangle, and the perpendicular EG divides the chord AB into two equal parts (III. 3.); the line FB will be equal to the difference of the segments AG , GF .

And, because E is the centre of the circle, and AE is equal to EI or EH , the line FI will be equal to the sum of the
the

the sides AE and EF ; and FH will be equal to their difference.

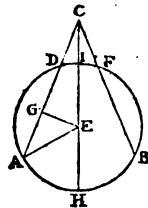
But the rectangle under the sum and difference of the two sides of any triangle, is equal to the rectangle under the base and the difference of the segments of the base (*Cor. II. 16.*); whence the rectangle of IF , FH is equal to the rectangle of AF , FB .

And, in the same manner, it may be proved, that the rectangle of IF , FH , is equal to the rectangle of DF , FC ; consequently the rectangle of AF , FB is also equal to the rectangle of DF , FC . Q.E.D.

SCHOLIUM. When the two lines intersect each other in the centre of the circle, the rectangles of their segments will manifestly be equal, because the segments themselves are all equal.

P R O P. XXVIII. THEOREM.

If two right lines be drawn from any point without a circle, to the opposite part of the circumference, the rectangle of the whole and external part of the one, will be equal to the rectangle of the whole and external part of the other.



Let $ADFB$ be a circle, and AC, BC any two right lines, drawn from the point C , to the opposite part of the circumference; then will the rectangle of AC, CD be equal to the rectangle of BC, CF .

For, through the centre E , and the point C , draw the right line CH ; and, from the point E , draw EG at right angles to AC (I. 12.) and join AE .

Then, since AEC is a triangle, and the perpendicular EG divides the chord AD into two equal parts (III. 3.), the line DC will be equal to the difference of the segments AG, GC .

And, because E is the centre of the circle, and AE is equal to EH , or EI , the line HC will be equal to the sum of the sides AE, EC , and IC will be equal to their difference.

But

But the rectangle under the sum and difference of the two sides of any triangle, is equal to the rectangle under the base and the difference of the segments of the base (*Cor. II. 16.*); whence the rectangle of HC , CI is equal to the rectangle of AC , CD .

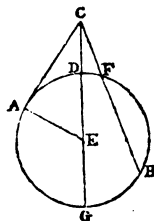
And, in the same manner, it may be proved, that the rectangle of HC , CI is also equal to the rectangle of CB , CF : consequently the rectangle of AC , CD will be equal to the rectangle of CB , CF .

Q.E.D.



P R O P. XXIX. THEOREM.

If two right lines be drawn from any point without a circle, the one to cut it, and the other to touch it; the rectangle of the whole and external part of the one, will be equal to the square of the other.



Let CB , CA be any two right lines drawn from the point C , the one to cut the circle $ADBG$, and the other to touch it; then will the rectangle of CB , CF be equal to the square of CA .

For,

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For, find E , the centre of the circle (III. 1.), and through the points E, C draw the line CEG ; and join EA :

Then, since AC is a tangent to the circle, and EA is a line drawn from the centre to the point of contact, the angle CAE is a right angle (III. 12.)

And, because EA is equal to EC , or ED , the line CG will be equal to the sum of EA, EC , and CD will be equal to their difference.

Since, therefore, the rectangle under the sum and difference of any two lines is equal to the difference of their squares (II. 13.), the rectangle of CG, CD will be equal to the difference of the squares of CE, EA .

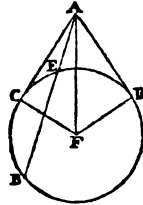
But the difference of the squares of CE, EA is equal to the square of CA (Cor. II. 14.); therefore the rectangle of CG, CD will also be equal to the square of CA .

And it has been shewn, in the last proposition, that the rectangle of CG, CD is equal to the rectangle of CB, CF ; consequently the rectangle of CB, CF , will also be equal to the square of CA . Q.E.D.

P R O P.

P R O P. XXX. THEOREM.

If two right lines be drawn from a point without a circle, the one to cut it, and the other to meet it; and if the rectangle of the whole and external part of the one be equal to the square of the other, the latter will be a tangent to the circle.



Let AB , AC be two right lines, drawn from any point A , without the circle CBD , the one to cut it, and the other to meet it; then, if the rectangle of AB , AE be equal to the square of AC , the line AC will be a tangent to the circle.

For, let F be the centre; and from the point A draw AD to touch the circle at D (III. 10.); also join FD , FA , FC .

Then, since AD is a tangent to the circle, and AEB cuts it, the rectangle of AB , AE is equal to the square of AD (III. 29.)

But the rectangle of AB , AE is also equal to the square of AC (*by Hyp.*); whence the square of AC is equal to the square of AD , or AC equal to AD (II. 3.)

1

And,

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And, because FC is equal to FD , AC to AD , and AF common to each of the triangles AFC , AFD , the angle ACF will also be equal to the angle ADF (I. 21.)

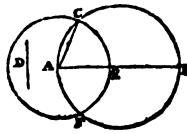
But, since AD touches the circle, and DF is a line drawn from the centre to the point of contact, the angle ADF is a right angle (III. 12.)

The angle ACF , therefore, is also a right angle; and CF produced is a diameter of the circle.

And since a right line, drawn from the end of the diameter, at right angles to it, touches the circle (III. 10.), AC will be a tangent to the circle CBD , as was to be shewn.

PROP. I. PROBLEM.

To place a right line in a given circle, equal to a given right line, not greater than the diameter of the circle.



Let ABC be a given circle, and D a given right line, not greater than the diameter; it is required to place a line in the circle ABC that shall be equal to D.

Find the centre of the circle (III. 1.), and draw any diameter AB; then if AB be equal to D the thing required is done.

But if not, make AE equal to D (I. 3.); and from the point A , at the distance AE , describe the circle FEC , cutting the former in C .

Join the points A, C; and AC will be equal to D, as was required.

For since A is the centre of the circle CEF, AC is equal to AE.

But D is also equal to AE , by construction; whence AC is, likewise, equal to D .

In the circle ABC , therefore, a right line has been placed equal to D , which was to be done.

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5. A circle is said to be inscribed in a rectilineal figure, when its circumference touches every side of that figure.



6. A circle is said to be described about a rectilineal figure, when its circumference passes through all the angular points of that figure.



7. A right line is said to be placed, or applied, in a circle, when the extremities of it are in the circumference of the circle.



8. All plane figures contained under more than four sides are called polygons; and if the angles, as well as sides, are all equal, they are called regular polygons.

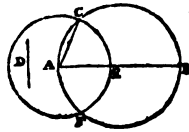


9. Polygons of five sides, are called pentagons; those of six sides hexagons; those of seven heptagons; and so on.

P R O P.

PROP. I. PROBLEM.

To place a right line in a given circle, equal to a given right line, not greater than the diameter of the circle.



Let ABC be a given circle, and D a given right line, not greater than the diameter; it is required to place a line in the circle ABC that shall be equal to D.

Find the centre of the circle (III. 1.), and draw any diameter AB; then if AB be equal to D the thing required is done.

But if not, make AE equal to D (I. 3.); and from the point A, at the distance AE, describe the circle FEC, cutting the former in C.

Join the points A, C; and AC will be equal to D, as was required.

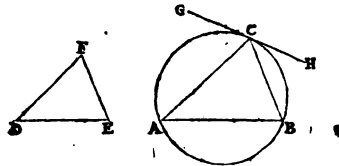
For since A is the centre of the circle FEC, AC is equal to AE.

But D is also equal to AE, by construction; whence AC is, likewise, equal to D.

In the circle ABC, therefore, a right line has been placed equal to D, which was to be done.

P R O P. II. PROBLEM.

To inscribe a triangle in a given circle, that shall be equiangular to a given triangle.



Let ABC be the given circle, and DEF the given triangle; it is required to inscribe a triangle in the circle ABC, that shall be equiangular to the triangle DEF.

Draw the right line GH to touch the circle ABC in the point C (III. 10.); and, make the angle HCB equal to the angle D (I. 20.), and the angle GCA to the angle E; and join AB; then will ACB be the triangle required.

For, since the right line GH is a tangent to the circle, and CB is a chord drawn from the point of contact, the angle HCB will be equal to the angle CAB in the alternate segment (III. 24.)

But the angle HCB is equal to the angle D, by construction; therefore the angle CAB is also equal to the angle D.

And, in the same manner, it may be proved, that the angle CBA is equal to the angle E.

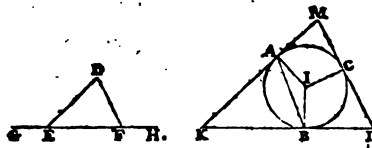
But, since the angle CAB is equal to the angle D, and the angle CBA to the angle E, the remaining angle ACB will also be equal to the remaining angle F (Cor. I. 28.), and consequently the triangle ACB is equiangular to the triangle DEF.

Q. E. D.

PROP.

P R O P. III. PROBLEM.

To circumscribe a triangle about a given circle, that shall be equiangular to a given triangle.



Let ABC be the given circle, and DEF the given triangle; it is required to circumscribe a triangle about the circle ABC that shall be equiangular to the triangle DEF .

Produce the line EF to G and H ; and, at the centre I , make the angles AIG, BIC equal to the angles DEG, DFH (I. 20.); and draw the lines MK, KL, LM , to touch the circle in the points A, B, C (III. 10.); and join AB .

Then, since the angles IAK, KBI , are, each of them, a right angle (III. 12.), the angles BAK, KBA , taken together, will be less than two right angles.

But when a right line intersects two other right lines, and makes the two interior angles, on the same side, together less than two right angles, those lines will, if produced, meet each other (I. 25. Cor.)

The line MK , therefore, meets the line KL ; and, if A, C, B be joined, the same may be proved of the lines KL, LM and MK ; consequently the figure KLM is a triangle.

And, because the four angles of the quadrilateral $AIBK$ are equal to four right angles (Cor. I. 28.), and the angles

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IAK, KBI are each a right angle, the remaining angles AIB, BKA will be equal to two right angles.

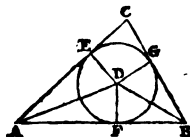
But the angles DEG, DEF are also equal to two right angles (I. 13.); therefore, since the angle DEG is equal to the angle AIB (*by Const.*), the remaining angle BKA will be equal to the remaining angle DEF.

And, in the same manner, it may be proved, that the angle CLB is equal to the angle DFE.

The angle MKL, therefore, being equal to the angle DEF, and the angle MLK to the angle DFE, the remaining angle KML will also be equal to the remaining angle EDF; and consequently the triangle KLM is equiangular to the triangle EFD. Q. E. D.

P R O P. IV. PROBLEM.

In a given triangle to inscribe a circle,



Let ABC be the given triangle; it is required to inscribe a circle in it.

Bisect the angles CAB, ABC, with the right lines AD, DB (I. 9.)

Then, since the angles DAB, DBA are less than two right angles (I. 28.), the lines AD, DB, will, if produced, meet each other (I. 25. *Cor.*)

And, if from the point of intersection D, there be drawn the perpendiculars DF, DG and DE, they will be the radii of the circle required.

For,

For, since the angle EAD is equal to the angle DAF (*by Const.*), and the angle AED to the angle DFA , (being each of them right angles), the remaining angle EDA will also be equal to the remaining angle ADF (I. 28. *Cor.*)

The triangles ADE , DAF , therefore, being equiangular, and having the side AD common to both, the side DE will also be equal to the side DF (I. 21.)

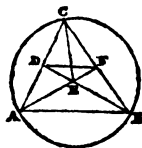
And, in the same manner, it may be proved, that the side DG is equal to the side DF .

The right lines DE , DG and DF are, therefore, all equal to each other; and the angles at the points F , E and G are right angles, by construction.

If, therefore, a circle be described from the centre D , with either of the distances DE , DG or DF , it will touch the sides in the points E , G , F (III. 10.) and be inscribed in the triangle ABC , as was to be done.

P R O P. V. PROBLEM.

To circumscribe a circle about a given triangle.



Let ABC be the given triangle; it is required to circumscribe a circle about it.

Bisect the sides AC , CB with the perpendiculars DE , EF (I. 10 and 11.); and join DF .

I 4

Then,

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Then, since the angles EDF, DFE are less than two right angles (*by Conf.*), the lines DE, EF will meet each other (I. 25. *Cor.*)

Let E, therefore, be their point of intersection, and draw the lines EA, EC and EB.

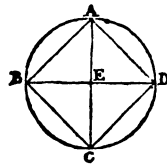
Then, because AD is equal to DC (*by Conf.*), DE common, and the angle ADE equal to the angle EDC (being each of them right angles), the base AE will also be equal to the base EC (I. 4.)

And, in the same manner, it may be proved, that EC is equal to EB; consequently EA, EC and EB are all equal to each other.

If, therefore, a circle be described from the point E, at either of the distances EA, EC or EB, it will pass through the remaining points, and circumscribe the triangle ABC, as was to be done.

P R O P. VI. PROBLEM.

To inscribe a square in a given circle,



Let ABCD be the given circle; it is required to inscribe a square in it.

Through E, the centre of the circle, draw any two diameters AC, BD at right angles to each other (I. 11, 12.), and join AB, BC, CD and DA; then will ABCD be the square required.

For

For since the two sides BE , EA , are equal to the two sides ED , EA , and the angle BEA to the angle AED , (being each of them right angles), the base BA will be equal to the base AD (I. 4.)

And, in the same manner, it may be proved, that the sides BC , CD are each equal to the sides BA , AD ; whence the figure $BCDA$ is equilateral.

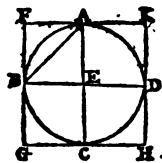
It is also rectangular: for since BDA is a semi-circle, the angle BAD is a right angle (III. 12.)

And, for the same reason, the angles ABC , BCD and CDA are each of them right angles.

The figure $BCDA$, therefore, being equilateral, and having all its angles right angles is a square, and it is inscribed in the circle $ABCD$, as was to be done.

P R O P. VII. PROBLEM.

To circumscribe a square about a given circle.



Let $ABCD$ be the given circle; it is required to circumscribe a square about it.

Draw any two diameters AC , BD at right angles to each other (I. 11, 12.); and through the points A , B , C , D , draw the tangents KF , FG , GH , HK (III. 10.); and join AB .

Then,

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5. A circle is said to be inscribed in a rectilineal figure, when its circumference touches every side of that figure.



6. A circle is said to be described about a rectilineal figure, when its circumference passes through all the angular points of that figure.



7. A right line is said to be placed, or applied, in a circle, when the extremities of it are in the circumference of the circle.



8. All plane figures contained under more than four sides are called polygons; and if the angles, as well as sides, are all equal, they are called regular polygons.

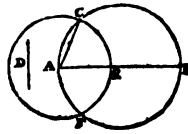


9. Polygons of five sides, are called pentagons; those of six sides hexagons; those of seven heptagons; and so on.

P R O P.

PROP. I. PROBLEM.

To place a right line in a given circle, equal to a given right line, not greater than the diameter of the circle.



Let ABC be a given circle, and D a given right line, not greater than the diameter; it is required to place a line in the circle ABC that shall be equal to D .

Find the centre of the circle (III. 1.), and draw any diameter AB ; then if AB be equal to D the thing required is done.

But if not, make AE equal to D (I. 3.); and from the point A , at the distance AE , describe the circle FEC , cutting the former in C .

Join the points A, C ; and AC will be equal to D , as was required.

For since A is the centre of the circle CEF , AC is equal to AE .

But D is also equal to AE , by construction; whence AC is, likewise, equal to D .

In the circle ABC , therefore, a right line has been placed equal to D , which was to be done.

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And if, to each of these angles, there be added the angle FBD , the whole angle DBE or FEB will be equal to the angles FDB , FBD , taken together.

But the angle DBE is equal to the angle DEB , or FEB , (I. 5.), and the angles FDB , FBD to the angle EFB (I. 28.); whence the angle FEB will be equal to the angle EFB , and the side EB to the side BF (I. 5.)

And since EB is equal to FD , by construction, BF will also be equal to FD , and the angle FDB to the angle FBD (I. 5.)

These two angles, therefore, taken together, are double the angle FDB ; whence the angle EFB , or its equal FEB , is also double the angle FDB .

But the angle FEB , or CEB , is equal to the angle CAB (III. 15.), and the angle FDB , or EDB , to the angle ACB (III. 14. and I. Ax. 6.); consequently the angle CAB is also double the angle ACB .

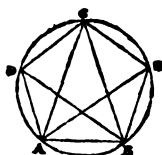
And, since EAC , EBC are right angled triangles (III. 16.), having EA equal to EB (*by Constr.*) and EC common, the remaining side AC will be equal to the remaining side CB (III. 8. Cor.)

The triangle ABC , therefore, is isosceles, and has each of the angles at its base double the angle at the vertex; as was to be shewn.

PROP.

PROP. XI. PROBLEM.

In a given circle to inscribe a regular pentagon.



Let $CDARE$ be the given circle; it is required to inscribe a regular pentagon in it.

Make the isosceles triangle ABC such, that each of the angles CAB , CBA may be double the angle ACB (IV. 10.)

Bisect the angles CAB , CBA with the lines AE , BD (I. 9.), and join the points AD , DC , CE , EB ; then will $ABECD$ be the pentagon required.

For, since the angles CAB , CBA are each double the angle ACB (*by Const.*), and the lines AE , BD bisect them, the angles ACB , CAE , EAB , ABD and DBC are all equal to each other.

And since equal angles stand upon equal circumferences (III. 21.), the arcs CD , DA , AB , BE and EC are also equal to each other.

But equal arcs are subtended by equal chords (III. 22.); consequently the sides CD , DA , AB , BE and EC are, likewise, equal.

The figure $ABECD$ is, therefore, equilateral: and it is also equiangular.

For, since the arc CD is equal to the arc BE , to each of them add DAB , and the arc $CDAB$ will be equal to the arc $DABE$.

But

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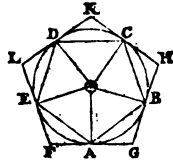
But equal angles are subtended by equal arcs (III. 21.), whence the angle CEB is equal to the angle DCE .

And, in the same manner, it may be shewn, that each of the angles CDA , DAB , ABE are equal to the angle CEB or DCE .

The pentagon $ABECD$, therefore, is both equilateral and equiangular, and it is inscribed in the given circle, as was to be done.

P R O P. XII.

About a given circle to describe a regular pentagon.



Let $ABCDE$ be the given circle; it is required to circumscribe it with a regular pentagon.

Inscribe the regular pentagon $DEABC$ (IV. 11.), and through the points A , B , C , D , E , draw the tangents FG , GH , HK , KL and LF ; also join the points O, A , O, B , O, C , O, D and O, E .

Then, since the angles OEF , OAF are right angles (III. 12.), the angles OEA , OAE , taken together, are less than two right angles; whence the lines LF , FG will meet each other (I. 25. *Cor.*)

And, in the same manner, it may be proved, that all the other lines FG , GH , HK , KL and LF will meet each other.

And,

And, since OE , OA and OB are all equal to each other, and EA is equal to AB , the angles OEA , OAE , OAB and OBA will be all equal to each other (I. 7.)

But the angles at the points E , A , B are also equal, being each of them right angles (III. 12.); consequently the angles AEF , EAF , BAG and ABG are likewise equal; and the angle F equal to the angle G (I. 21.)

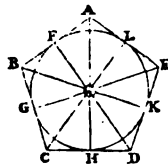
And, in the same manner, it may be shewn, that the angles G , H , K , L and F are all equal to each other.

Since, therefore, the triangles EFA , AGB , &c. are equiangular, and have their bases EA , AB , &c. equal to each other, the remaining sides EF , FA , AG , GB , &c. will also be equal (I. 21.):

And since LF , FG , &c. are the doubles of EF , FA , &c. the figure $FGHKL$ is a regular pentagon; and it is circumscribed about the circle $ABCDE$, as was to be done.

P R O P. XIII. PROBLEM.

In a given regular pentagon to inscribe a circle.



Let $ABCDE$ be the given regular pentagon; it is required to inscribe a circle in it.

Bisect any two angles BCD , CDE with the right lines CO , OD (I. 9.), and the point of intersection O will be the centre of the circle required.

K

For

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For draw the lines OB , OA and OE , and let fall the perpendiculars OH , OK , OL , OF and OG (I. 12.):

Then, because CB is equal to CD (*by Hyp.*), CO common, and the angle BCO equal to the angle OCD (*by Conf.*), the angle CBO will also be equal to the angle ODC (I. 7.)

But the angle ODC is equal to half the angle CDE (*by Conf.*), and the angle CDE is equal to the angle CBA (*by Hyp.*); consequently the angle CBO is also equal to half the angle CBA .

The angle CBA , therefore, is bisected by the line BO ; and, in the same manner, it may be shewn, that the angles at the points A , E are bisected, by the lines AO , OE .

Again, because the triangles OGC , OCH are equiangular, and have OC common to each, the perpendicular OG will be equal to the perpendicular OH (I. 21.)

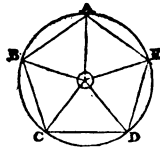
And, in the same manner, it may be shewn, that OH , OK , OL , OF and OG are all equal to each other.

If, therefore, a circle be described from the centre O , at either of these distances, it will pass through the remaining points, and be inscribed in the pentagon $ABCDE$, as was to be done.

PROP,

P R O P. XIV. PROBLEM.

To describe a circle about a given regular pentagon,



Let $ABCDE$ be a given regular pentagon ; it is required to circumscribe it with a circle.

Bisect any two angles BCD , CDE , with the right lines CO , OD (I. 9.), and the point of intersection O will be the centre of the circle required,

For, draw the lines OB , OA and OE :

Then, because CB is equal to CD (*by Hyp.*), CO common, and the angle BCO equal to the angle OCD (*by Const.*), the angle CBO will also be equal to the angle ODC (I. 4.)

But the angle ODC is equal to half the angle CDE , (*by Const.*), and the angle CDE is equal to the angle CBA (*by Hyp.*) ; whence the angle CBO is also equal to half the angle CBA .

The angle CBA , therefore, is bisected by the line BO ; and, in the same manner, it may be shewn, that the angles at the points A , E are bisected, by the lines AO , OE .

Since, therefore, the angle OCD is equal to the angle ODC (*by Hyp. and Ax. 7.*), the side OC will also be equal to the side OD (I. 5.)

And, in the same manner, it may be shewn, that OD , OE , OA , OB and OC are all equal to each other.

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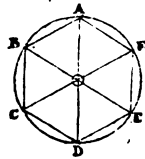
If,

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If, therefore, a circle be described from the point O , at either of these distances, it will pass through the remaining points, and circumscribe the pentagon $ABCDE$, as was to be done,

P R O P. XV. PROBLEM.

In a given circle to inscribe a regular hexagon,



Let $ACEF$ be a given circle; it is required to inscribe a regular hexagon in it.

Through the centre O draw the diameter AD , and make DC equal to DO (IV. I.), and it will be the side of the hexagon required.

For, draw the diameter CF , and make BE parallel to CD (I. 27.); and join the points DE , EF , FA , AB and BC :

Then, since DOC is an equilateral triangle, the angles ODC , OCD and DOC will be all equal to each other (I. 5. Cor.)

And, because OE is parallel to CD , the angle EOD will be equal to the angle ODC (I. 24.), and the angle FOE to the angle OCD (I. 25.)

But the angles ODC , OCD are each equal to the angle DOC ; therefore, the angles DOC , EOD and FOE are all equal to each other; as are also the opposite angles FOA , AOB and BOC .

Since,

Since, therefore, the triangles COB , DOE , &c. have two sides, and the included angle of the one equal to two sides and the included angle of the other, they will be equal in all respects (I. 4.)

The sides CD , DE , EF , &c. are therefore all equal to each other, as are also the angles BCD , CDE , &c. whence $ABCDEF$ is a regular hexagon; and it is inscribed in the circle $ACEF$, as was to be done.

SCHOLIUM. Besides the figures here constructed, and those arising from thence by continual bisections, or taking the differences, no other regular polygon can be described, by any known method, purely geometrical.

It may also be observed that some of these figures, as well as several others, in the former part of the work, may often be described in a much easier way, for practical purposes; but the principles upon which they depend can only be obtained from the following books of the Elements.

BOOK V.

DEFINITIONS.

1. A less magnitude is said to be a part of a greater, when the less is contained a certain number of times in the greater.

2. A greater magnitude is said to be a multiple of a less, when the greater is equal to a certain number of times the less.

3. Ratio is a certain mutual relation of two magnitudes of the same kind, which arises from considering the quantity of each.

4. When four magnitudes are compared together, the first and third are called the antecedents, and the second and fourth the consequents.

5. Four magnitudes are said to be proportional, when any equimultiples whatever of the antecedents, are, each of them, either equal to, greater, or less, than any equimultiples whatever of their consequents.

6. Inverse ratio is, when the consequents are made the antecedents, and the antecedents the consequents.

7. Alternate ratio is, when antecedent is compared with antecedent, and consequent with consequent.

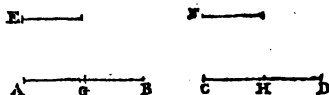
8. Compounded ratio is, when each antecedent and its consequent, taken as one quantity, is compared, either with the consequents, or the antecedents.

9. Divided ratio is, when the difference of each antecedent and its consequent, is compared, either with the consequents, or the antecedents.

PROP.

P R O P. I. THEOREM.

If any number of magnitudes be equimultiples of as many others, each of each; whatever multiple any one of them is of its part, the same multiple will all the former be of all the latter.



Let any number of magnitudes AB , CD be equimultiples of as many others E , F , each of each; then whatever multiple AB is of E , the same multiple will AB and CD together, be of E and F together.

For since AB is the same multiple of E that CD is of F (*by Hyp.*), as many magnitudes as there are in AB equal to E , so many will there be in CD equal to F .

Divide AB into magnitudes equal to E (I. 35.), which let be AG , GB ; and CD into magnitudes equal to F , which let be CH , HD .

Then the number of magnitudes CH , HD , in the one, will be equal to the number of magnitudes AG , GB , in the other.

And because AG is equal to E , and CH to F (*by Const.*), AG and CH , taken together, will be equal to E and F taken together.

For the same reason, because GB is equal to E , and HD to F , GB and HD taken together, will be equal to E and F taken together.

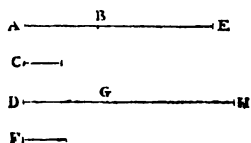
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As many magnitudes, therefore, as there are in AB equal to E, so many are there in AB and CD together, equal to E and F together.

And, consequently, whatever multiple AB is of E, the same multiple will AB and CD together be of E and F together. Q.E.D.

P R O P. II. THEOREM.

If any number of magnitudes be multiples of the same magnitude, and as many others be the same multiples of another magnitude, each of each, the sum of all the former will be the same multiple of the one, as the sum of all the latter is of the other.



Let any number of magnitudes AB, BE, be multiples of the same magnitude C, and as many others DG, GH, the same multiples of another F, each of each; then will the whole AE, be the same multiple of C, as the whole DH, is of F.

For since AB is the same multiple of C that DG is of F (*by Hyp.*), there will be as many magnitudes in AB equal to C, as there are in DG equal to F.

And because BE is the same multiple of C that GH is of F (*by Hyp.*), there will be as many magnitudes in BE equal to C, as there are in GH equal to F.

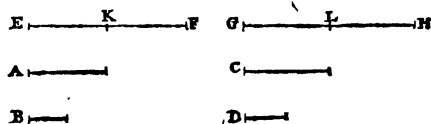
As

As many magnitudes, therefore, as there are in the whole AE equal to C , so many will there be in the whole DH equal to F .

The whole AE , therefore, is the same multiple of C , as the whole DH is of F . Q.E.D.

P R O P. III. THEOREM.

If the first of four magnitudes be the same multiple of the second as the third is of the fourth; and if of the first and third there be taken equimultiples, these will also be equimultiples, the one of the second, and the other of the fourth.



Let A the first, be the same multiple of B the second, as C the third, is of D the fourth; and let EF and GH be equimultiples of A and C ; then will EF be the same multiple of B , that GH is of D .

For since EF is the same multiple of A that GH is of C (*by Hyp.*), there will be as many magnitudes in EF equal to A , as there are in GH equal to C .

Divide EF into the magnitudes EK , KF each equal to A (I. 35.); and GH into the magnitudes GL , LH , each equal to C .

Then

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Then will the number of magnitudes EK, KF in the one, be equal to the number of magnitudes GL, LH in the other.

And because A is the same multiple of B that C is of D (by Hyp.), and EK is equal to A, and GL to C (by Const.), EK will be the same multiple of B, that GL is of D.

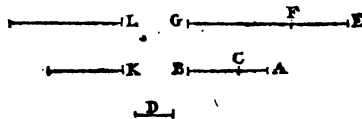
In like manner, since KF is equal to A, and LH to C, KF will be the same multiple of B, that LH is of D.

And since EK, KF are each multiples of B, and GL, LH are each the same multiples of D, the whole EF will be the same multiple of B, as the whole GH is of D (V. 2.)

Q.E.D.

P R O P. IV. THEOREM.

If the first of three magnitudes be greater than the second, and the third be any magnitude whatever, some equimultiples of the first and second may be taken, and some multiple of the third such, that the former shall be greater than that of the third, but the latter not greater.



Let AB, BC be two unequal magnitudes, and D any other magnitude whatever; then there may be taken some equimultiples of AB, BC, and some multiple of D such, that

that the multiple of AB shall be greater than that of D , but the multiple of BC not greater.

For of BC , CA take any equimultiples GF , FE such, that they may be each greater than D ; and of D take the multiples K and L such, that L may be that which is first greater than GF , and K that which is next less than L .

Then, because L is that multiple of D which is the first that becomes greater than GF , the next preceding multiple K will not be greater than GF ; that is GF will not be less than K .

And, since FE is the same multiple of AC that GF is of BC (*by Const.*); GF will also be the same multiple of BC that EG is of AB (V. 1.)

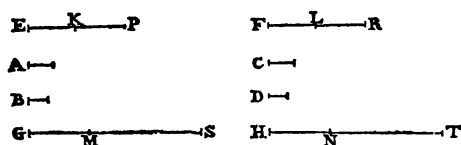
The magnitudes EG and GF are, therefore, equimultiples of the magnitudes AB and BC , and L is a multiple of D .

And, since GF is not less than K , and EF is greater than D (*by Const.*), the whole EG will be greater than K and D taken together.

But K and D , taken together, are equal to L (*by Const.*); therefore EG will be greater than L , and FG not greater than L , as was to be shewn.

P R O P. V. THEOREM.

If four magnitudes be proportional, any equimultiples whatever of the antecedents will be proportional to any equimultiples whatever of the consequents.



Let A be to B as C is to D, and of A and C take any equimultiples EK, FL; and of B and D any equimultiples GM, HN; then will EK be to GM, as FL is to HN.

For of EK and FL take any equimultiples whatever EP, FR; and of GM and HN any equimultiples whatever GS, HT:

Then, since EK is the same multiple of A, that FL is of C (*by Const.*), and of EK, FL have been taken the equimultiples EP, FR, EP will be the same multiple of A, that FR is of C (V. 3.)

And, in the same manner, it may be shewn, that GS is the same multiple of B, that HT is of D.

But A has the same ratio to B that C has to D (*by Hyp.*); and of A and C have been taken the equimultiples EP, FR; and of B and D the equimultiples GS, HT.

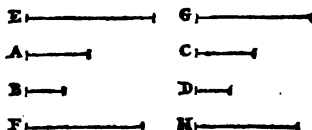
If, therefore, EP be greater than GS, FR will also be greater than HT; and if equal, equal; and if less, less (V. Def. 5.)

And,

And, since EP , FR are any equimultiples whatever of EK , FL ; and GS , HT are any equimultiples whatever of GM , HN ; EK will have the same ratio to GM , that FL has to HN (*V. Def. 5.*) Q.E.D.

P R O P. VI. THEOREM.

If four magnitudes be proportional, and the first be greater than the second, the third will also be greater than the fourth; and if equal, equal; and if less, less.



Let A have to B the same ratio that c has to D ; then if A be greater than B , c will also be greater than D ; and if equal, equal; and if less, less.

For, of A and C take any equimultiples E and G , and of B and D the same equimultiples F and H .

Then, because A is to B , as c is to D (*by Hyp.*) if E be greater than F , G will also be greater than H ; and if equal, equal; and if less, less (*V. Def. 5.*)

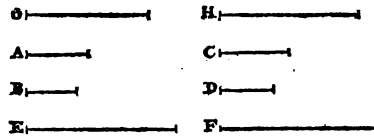
And, since E , F , G , H are the same multiples of A , B , c , D , each of each, these last magnitudes will also observe the same agreement of equality, excess, or defect with their equimultiples.

If, therefore, A be greater than B , c will also be greater than D ; and if equal, equal; and if less, less. Q.E.D.

P R O P.

P R O P. VII. THEOREM.

If four magnitudes be proportional, they will be proportional also when taken inversely.



If A has to B the same ratio that C has to D ; then, inversely, B will have to A the same ratio that D has to C .

For, of B and D take any equimultiples whatever E and F ; and of A and C any equimultiples whatever G and H :

Then, since A is to B as C is to D (*by Hyp.*), and G , H are equimultiples of A , C , and E , F of B , D (*by Const.*), if G be greater than E , H will be greater than F ; and if equal, equal; and if less, less (*V. Def. 5.*)

And, because G has with E the same agreement of equality, excess, or defect, that H has with F , E will have with G the same agreement of equality, excess, or defect, that F has with H .

If, therefore, E be greater than G , F will also be greater than H ; and if equal, equal; and if less, less.

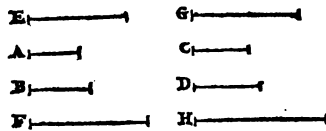
But E and F are any equimultiples whatever of B and D (*by Const.*); and G and H are any equimultiples whatever of A and C ; therefore, B is to A as D is to C (*V. Def. 5.*)

Q. E. D.

P R O P.

P R O P. VIII. THEOREM.

If the first of four magnitudes be the same multiple or part of the second as the third is of the fourth ; the first will have the same ratio to the second as the third has to the fourth,



Let A the first, be the same multiple of B the second, that C the third is of D the fourth ; then will A have to B the same ratio that C has to D.

For of A and C take any equimultiples whatever E and G ; and of B and D any equimultiples whatever F and H :

Then, because A is the same multiple of B that C is of D (*by Hyp.*), and E is the same multiple of A that G is of C (*by Const.*), E will also be the same multiple of B that G is of D (V. 3.)

And, since E is the same multiple of B that G is of D, and F is the same multiple of B that H is of D (*by Const.*), if E be greater than F, G will be greater than H ; and if equal, equal ; and if less, less.

But E and G are any equimultiples whatever of A and C ; and F and H are any equimultiples whatever of B and D ; therefore, A will have to B the same ratio that C has to D (V. Def. 5.)

Again,

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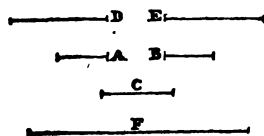
Again, let the first B, be the same part of the second A, as the third D, is of the fourth C; then will B have to A the same ratio that D has to C.

For A is the same multiple of B that C is of D (*by Hyp.*); therefore A will have to B the same ratio that C has to D (V. 8.)

And, since A is to B as C is to D, therefore, also, inverfely, B is to A as D is to C (V. 7.) Q.E.D.

P R O P. IX. THEOREM.

Equal magnitudes have the same ratio to the same magnitude, and the same has the same ratio to equal magnitudes.



Let A and B be equal magnitudes, and C any other magnitude whatever; then A will have to C the same ratio that B has to C.

For of A and B take any equimultiples whatever D and E; and of C any multiple whatever F:

Then, because D is the same multiple of A, that E is of B, and A is equal to B, D will also be equal to E.

And, since D and E are equal to each other, if D be greater than F, E will also be greater than F; and if equal, equal; and if less, less.

But

But D and E are any equimultiples whatever of A and B , and F is any multiple whatever of C ; therefore A is to e as B is to C (V. Def. 5.)

Again; let A and B be equal magnitudes, and C any other magnitude whatever; then C has to A the same ratio that it has to B .

For, having made the same construction as before, D may, in like manner, be shewn to be equal to E :

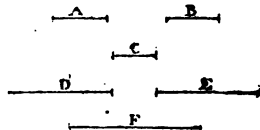
And, since D is equal to E , if F be greater than D , it will also be greater than E ; or if equal, equal; or if less, less.

But F is any multiple whatever of C , and D and E are any equimultiples whatever of A and B ; therefore, C is to A as C is to B (V. Def. 5.)

Q. E. D.

P R O P. X. THEOREM:

Magnitudes which have the same ratio to the same magnitude are equal to each other; and those to which the same magnitude has the same ratio are equal to each other.



Let A have to C the same ratio that B has to C ; then will A be equal to B .

For, if they be not equal, one of them must be greater than the other.

L

Let

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Let A be the greater; and of A and B take the equimultiples D and E , and of C the multiple F such, that D may be greater than F , and E not greater than F (V. 4.)

Then, since A is to C as B is to C , and D and E are equimultiples of A and B , and F is a multiple of C , D being greater than F , E will also be greater than F (V. Def. 5.)

But, by construction, E is not greater than F ; whence it is greater and not greater at the same time, which is absurd.

The magnitude A is, therefore, not greater than B ; and in the same manner it may be shewn that it is not less; consequently they are equal to each other.

Again, let C have to A the same ratio that it has to B ; then will A be equal to B .

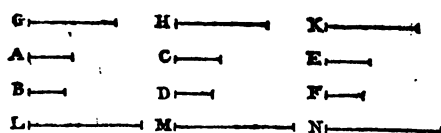
For, since C is to A as C is to B , therefore, also, inversely, A will be to C as B is to C (V. 7.)

But magnitudes which have the same ratio to the same magnitude have been shewn to be equal to each other, therefore A is equal to B . Q. E. D.

PROP.

P R O P. XI. THEOREM.

Ratios which are the same to the same ratio, are the same to each other.



Let A be to B as C is to D, and C to D as E is to F ; then will A be to B as E is to F.

For, of A, C and E take any equimultiples whatever G, H and K ; and of B, D and F any equimultiples whatever L, M and N :

Then, since A is to B as C is to D (*by Hyp.*), and G and H are equimultiples of A and C, and L and M of B and D, if G be greater than L, H will be greater than M ; and if equal, equal ; and if less, less (V. Def. 5.)

And, because C is to D as E is to F (*by Hyp.*), and H and K are equimultiples of C and E, and M and N of D and F ; if H be greater than M, K will be greater than N ; and if equal, equal ; and if less, less (V. Def. 5.)

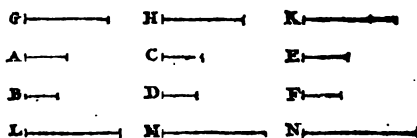
But if G be greater than L, it has been shewn that H will also be greater than M ; and if equal, equal ; and if less, less ; whence, if G be greater than L, K will also be greater than N ; and if equal, equal ; and if less, less.

And since G and K are any equimultiples whatever of A and E, and L and N are any equimultiples whatever of B and F, A will have to B the same ratio that E has to F (V. Def. 5.)

Q. E. D.

P R O P. XII. THEOREM.

If any number of magnitudes be proportional, either of the antecedents will be to its consequent, as the sum of all the antecedents is to the sum of all the consequents.



Let A be to B as C is to D, and as E is to F; then will A be to B, as A, C and E together, are to B, D and F together.

For, of A, C and E take any equimultiples whatever G, H and K; and of B, D and F any equimultiples whatever L, M and N:

Then, since A is to B, as C is to D (*by Hyp.*), and G, H are equimultiples of A, C, and L, M of B, D, if G be greater than L, H will be greater than M, and if equal, equal; and if less, less (*V. Def. 5.*)

And because A is also to B as E is to F (*by Hyp.*), and G, K are equimultiples of A, E, and L, N of B, F, if G be greater than L, K will be greater than N; and if equal, equal; and if less, less (*V. Def. V.*)

From hence it follows, that if G be greater than L, G, H and K together, will be greater than L, M and N together; and if equal, equal; and if less, less.

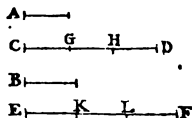
But

But C , and C, H, K together, are any equimultiples whatever of A , and A, C, E together (*by Const.*); and L , and L, M, N together, are any equimultiples whatever of B , and B, D, F together; whence, as A is to B , so is A, C and E together, to B, D and F together (*V. Def. 5.*)

Q. E. D.

P R O P. XIII. THEOREM.

Equimultiples of any two magnitudes have the same ratio as the magnitudes themselves.



Let CD be the same multiple of A that EF is of B ; then will CD have the same ratio to EF that A has to B .

For, since CD is the same multiple of A that EF is of B : there are as many magnitudes in CD equal to A , as there are in EF equal to B .

Let CD be divided into the magnitudes CG, GH, HD , each equal to A (*I. 25.*); and EF into the magnitudes EK, KL, LF , each equal to B .

Then, the number of magnitudes CG, GH, HD in the one, will be equal to the number of magnitudes EK, KL, LF in the other.

And, because DH, HG, GC are all equal to each other, as are also FL, LK, KE , DH will be to FL as HG to LK , and as GC to KE (*V. 9.*)

L 3

And,

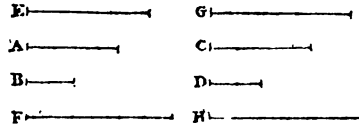
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And, since any antecedent is to its consequent; as all the antecedents are to all the consequents (V. 12.), FL will be to DH, as FE is to DC.

But DH is equal to A (*by Conf.*), and FL is equal to B; therefore B will be to A, as FE to DC; and, inverfely, DC to FE as A to B. Q. E. D.

P R O P. XIV. THEOREM.

If four magnitudes of the same kind be proportional, and the first be greater than the third, the second will also be greater than the fourth; and if equal, equal; and if less, less.



Let A be to B as C is to D; then if A be greater than C, B will also be greater than D; and if equal, equal; and if less, less.

First, let A be greater than C; then B will be greater than D.

For, of A, C take the equimultiples E, G, and of B the multiple F such, that E may be greater than F, but G not greater (V. 4.); and make H the same multiple of D that F is of B.

Then, because A is to B as C is to D (*by Hyp.*), and E, G are equimultiples of A, C, and F, H of B, D (*by Conf.*), E being greater than F, G will also be greater than H (V. 5.)

And,

And, since, by construction, f is not less than g , and g has been proved to be greater than h , f will likewise be greater than h .

But f and h are equimultiples of b and d (*by Const.*); therefore, since f is greater than h , b will also be greater than d .

Secondly, let A be equal to c : then will b be equal d .

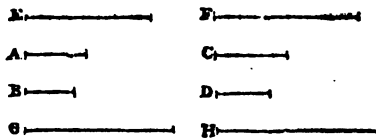
For, A is to b as c is to d (*by Hyp.*); or, since A is equal to c , A is to b as A is to d ; therefore b is equal to d (V. 10.)

Thirdly, let A be less than c ; then will b be less than d .

For, c is to d as A is to b , by the proposition; therefore c being greater than A , d will also be greater than b , by the first case. Q. E. D.

P R O P. XV. THEOREM.

If four magnitudes of the same kind be proportional, they will be proportional also when taken alternately.



Let A be to b as c is to d ; then, also, alternately, A will be to c as b is to d .

For, of A and b take any equimultiples whatever E and G ; and of c and d any equimultiples whatever F and H :

L 4

Then,

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Then, since E is the same multiple of A that G is of B , and that magnitudes have the same ratio as their equimultiples (V. 13.), A is to B as E is to G .

But A is to B as C is to D , by the proposition; whence C is to D as E is to G (V. 11.)

In like manner, because F is the same multiple of C that H is of D , C will be to D as F is to H (V. 13.)

But C has been shewn to be to D as E is to G ; consequently, E will be to G as F is to H (V. 11.)

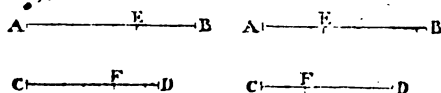
Since, therefore, E has the same ratio to G that F has to H , if E be greater than F , G will also be greater than H ; and if equal, equal; and if less, less (V. Def. 5.)

But E and G are any equimultiples whatever of A and B ; and F and H are any equimultiples whatever of C and D ; therefore A is to C as B is to D (V. Def. 5.)

Q. E. D.

P R O P, XVI. THEOREM,

If four magnitudes be proportional, the sum of the first and second, will be to the first or second, as the sum of the third and fourth, is to the third or fourth.



Let AE be to EB as CF is to FD ; then will AB be to BE , or AE , as CD is to DF , or CF .

For,

For, since AE is to EB as CF is to FD (*by Hyp.*); therefore, alternately, AE will be to CF as EB to FD (V. 15.)

And, since the antecedent is to its consequent as all the antecedents are to all the consequents (V. 12.), AE will be to CF as AB is to CD .

But ratios which are the same to the same ratio are the same to each other (V. 11.); whence AB will be to CD as EB is to FD ; and, alternately, AB to EB as CD to DF (V. 15.)

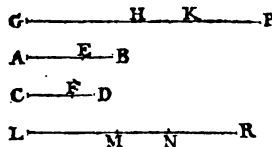
Again, since AE has been shewn to be to CF as AB is to CD , therefore, by alternation, AE will be to AB as CF is to CD (V. 15.)

But quantities which are directly proportional are also proportional when taken inversely; whence AB will be to AE as CD is to CF (V. 7.)

Q. E. D.

PROP. XVII. THEOREM.

If four magnitudes be proportional, the difference of the first and second, will be to the first or second, as the difference of the third and fourth, is to the third or fourth.



Let AB be to BE as CD is to DF ; then will AE be to AB , or EB , as CF is to CD , or FD .

For,

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For, of AE , EB , CF , FD take any equimultiples whatever GH , HK , LM , MN ; and of EB , FD any other equimultiples whatever KP , NR :

Then, because GH is the same multiple of AE that HK is of EB (*by Const.*), GH will be the same multiple of AE that GK is of AB (V. 1.)

But GH is the same multiple of AE that LM is of CF (*by Const.*); therefore GK is the same multiple of AB that LM is of CF .

In like manner, because LM is the same multiple of CF that MN is of FD (*by Const.*) LM will be the same multiple of CF that LN is of CD (V. 1.)

But LM has been shewn to be the same multiple of CF that GK is of AB ; therefore GK is the same multiple of AB that LN is of CD .

Again, because HK , KP are the same multiples of EB , that MN , NR are of FD (*by Const.*), HP will be the same multiple of EB , that MR is of FD (V. 2.)

And, since AB is to BE as CD to DF (*by Hyp.*), and GK , LN are equimultiples of AB , CD , and HP , MR of BE , DF , if GK be greater than HP , LN will be greater than MR ; and if equal, equal; and if less, less (V. Def. 5.)

From the two former of these take away the common part HK , and from the two latter, the common part MN ; then if GH be greater than KP , LM will be greater than NR , and if equal, equal; and if less, less.

But GH , LM are any equimultiples whatever of AE , CF (*by Const.*), and KP , NR are any equimultiples whatever of EB , FD ; whence AE is to CF as EB to FD (V. Def. 5.); and, alternately, AE to EB as CF to FD :

And,

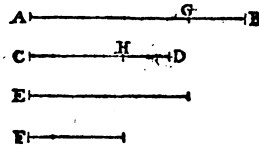
And, since AB is the sum of AE, EB, and CD of CF, FD, AB will be to AE or EB, as CD is to CF or FD (V. 16.); and, inversely, AE or EB to AB, as CF or FD to CD.

Q. E. D.

SCHOLIUM. When the consequents are greater than the antecedents, the same demonstration will hold, if the terms be taken inversely.

PROP. XVIII. THEOREM.

If four magnitudes of the same kind be proportional, the greatest and least of them, taken together, will be greater than the other two.



Let AB, CD, E, F be the four proportional magnitudes, of which AB is the greatest and F the least; then will AB and F together, be greater than CD and E together.

For in AB take AG equal to E, and in CD take CH equal to F (I. 3.)

Then, because AB is to CD as E to F (*by Hyp.*), and AG is equal to E and CH to F (*by Const.*), AB will be to CD as AG to CH.

But magnitudes which are proportional, are also proportional when taken alternately (V. 15.); therefore AB will be to AG as CD to CH.

And, since BG is the difference of AB and AG, and DH of CD and CH, BG will be to AB as DH to CD (V. 17.); and, inversely, AB to BG as CD to DH.

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But AB is greater than CD (*by Hyp.*); whence GB is also greater than HD (V. 14.)

And, because AG is equal to E , and CH to F , AG and F together, are equal to CH and E together.

To the first of these equals add GB , and to the second HD , then will AG , GB and F together, be greater than CH , HD and E together.

But AG , GB are equal to AB , and CH , HD to CD ; consequently AB and F together are greater than CD and E together. Q. E. D.

SCHOLIUM. That F must be the least of the four magnitudes when A is the greatest, appears from propositions VI. and XIII.

BOOK VI.

DEFINITIONS.

1. Similar rectilineal figures, are those which are equiangular, and have the sides about the equal angles proportional.

2. The homologous, or like sides, of similar figures, are those which are opposite to equal angles.

3. Two figures are said to have their sides reciprocally proportional, when the first consequent, and second antecedent, of the four terms, are both sides of the same figure.

4. Of three proportional quantities, the middle one is said to be a mean proportional between the other two; and the last a third proportional to the first and second.

5. Of four proportional quantities the last is said to be a fourth proportional to the other three, taken in order.

6. If any number of magnitudes be continually proportional, the ratio of the first and third is said to be duplicate that of the first and second; and the ratio of the first and fourth, triplicate that of the first and second.

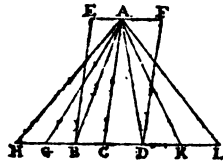
7. And of any number of magnitudes, of the same kind, taken in order, the ratio of the first to the last, is said to be compounded of the ratio of the first to the second, of the second to the third, and so on, to the last.

8. A right line is said to be divided in extreme and mean ratio, when the whole line is to the greater segment, as the greater segment is to the less.

PROP.

P R O P. I.

Triangles and parallelograms, having the same altitude, are to each other as their bases.



Let the triangles ABC , ACD , and the parallelograms EC , CF have the same altitude, or be between the same parallels BD , EF ; then will the base BC be to the base CD , as the triangle ABC is to the triangle ACD , or as the parallelogram EC is to the parallelogram CF .

For, in BD produced, take any number of parts whatever BG , GH , each equal to BC ; and DK , KL , any number whatever, each equal to CD ; and join AG , AH , AK and AL :

Then, because CB , BG , GH are all equal to each other, the triangles AHG , AGB , ABC will also be equal to each other (II. 5.); and whatever multiple the base HC is of the base BC , the same multiple will the triangle AHC be of the triangle ABC .

And, for the same reason, whatever multiple the base LC is of the base CD , the same multiple will the triangle ALC be of the triangle ADC .

If therefore, the base HC be equal to the base CL , the triangle AHC will be equal to the triangle ALC ; and if greater, greater; and if less, less.

But

But the base HC , and the triangle AHC , are any equimultiples whatever of the base BC , and the triangle ABC ; and the base CL and the triangle ALC are any equimultiples whatever of the base CD and the triangle ADC ; whence the base BC is to the base CD , as the triangle ABC is to the triangle ADC (V. Def. 5.)

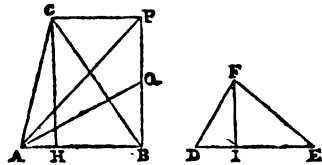
Again, because the parallelogram CE is double the triangle ABC (I. 32.), and the parallelogram CF is double the triangle ADC , the triangle ABC will be to the triangle ADC as the parallelogram CE is to the parallelogram CF (V. 12.)

But, it has been shewn, that the base BC is to the base CD , as the triangle ABC is to the triangle ADC ; therefore the base BC is also to the base CD , as the parallelogram CE is to the parallelogram CF . Q. E. D.

COROLL. Triangles and parallelograms, having equal altitudes, are to each other as their bases.

P R O P. II.

Triangles and parallelograms, having equal bases, are to each other as their altitudes.



Let ABC , DEF be two triangles, having the equal bases AB , DE , and whose altitudes are CH , FI ; then will the triangle

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triangle ABC have the same ratio to the triangle DEF , as CH has to FI .

For make BP perpendicular to AB , and equal to CH (I. 11 and 3.); and in BP take BQ equal to FI , and join AP , AQ and CP .

Then, because BP is equal to CH , and the base AB is common, the triangle ABP will be equal to the triangle ABC (II. 5.)

And, because AB is equal to DE , and BQ to FI ; the triangle ABQ will also be equal to the triangle DEF (II. 5.)

But the triangle ABP is to the triangle ABQ as BP is to BQ (VI. 1.); therefore the triangle ABC is also to the triangle DEF as BP is to BQ , or as CH to FI (V. 9.)

And, since parallelograms, having the same bases and altitudes, are the double of these triangles, they will, likewise, have to each other the same ratio as their altitudes.

Q. E. D.

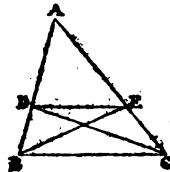
COR. 1. If the bases of equal triangles are equal, the altitudes will also be equal; and if the altitudes are equal, the bases will be equal.

COR. 2. From this, and the former proposition, it also appears, that rectangles, which have one side in each equal, are proportional to their other sides.

PROP.

PROP. III. THEOREM.

If a right line be drawn parallel to one of the sides of a triangle, it will cut the other sides proportionally : and if the sides be cut proportionally, the line will be parallel to the remaining side of the triangle.



Let ABC be a triangle, and DE be drawn parallel to the side BC ; then will AD be to DB , as AE is to EC .

For join the points B , E , and C , D :

Then, because the triangles DBE , DCE are upon the same base DE , and between the same parallels DE , BC , they will be equal to each other (I. 31.)

And, since equal magnitudes have the same ratio to the same magnitude (V. 9.), the triangle DBE will be to the triangle DAE , as the triangle DCE is to the triangle DAE .

But triangles of the same altitude are to each other as their bases (VI. 1.) ; whence the triangle DBE will be to the triangle DAE as DB is to DA .

For the same reason, the triangle DCE will be to the triangle DAE , as EC is to EA .

M

And,

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And, since ratios which are the same to the same ratio, are the same to each other (V. 11.) DB will be to DA as EC is to EA ; or, inversely, AD to DB as AE to EC .

Again, let the sides AB , AC be cut proportionally, in the points D and E ; then will the line DE be parallel to BC .

For, the same construction being made as before, the triangle DEB will be to the triangle DEA , as DB is to DA (VI. 1.); and the triangle EDC to the triangle DEA as EC to EA (VI. 1.)

And, since DB is to DA as EC is to EA (*by Const.*), the triangle DEB will be to the triangle DEA as the triangle EDC is to the triangle DEA (V. 11.)

But magnitudes which have the same ratio to the same magnitude are equal to each other (V. 10.); whence the triangle DEB is equal to the triangle EDC .

And since these triangles are equal to each other, and are upon the same base DE , they will have equal altitudes (VI. 2. *Cor.*), or stand between the same parallels; whence DE is parallel to BC .

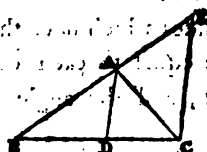
Q. E. D.

COROLL. In the same manner it may be shewn, that the sides of the triangle are proportional to any two of the parts into which they are divided; and that the like parts of each are also proportional.

PROP.

PROPOSITION IV. THEOREM.

If the vertex angle of a triangle be bisected, the segments of the base will have the same ratio with the other two sides: and if the segments have the same ratio with the other two sides the angle will be bisected.



Let the angle BAC of the triangle ABC be bisected, by the right line AD ; then will BD be to DC as BA is to AC .

For through the point C draw CE parallel to DA (I. 27.), and let BA be produced to meet CE in E :

Then, because the right line AC cuts the two parallel right lines AD , EC , the angle ACE will be equal to the alternate angle CAD (I. 24.)

But the angle CAD is equal to the angle BAD , by the proposition; therefore the angle BAD is also equal to the angle ACE .

And, in like manner, because the right line BE cuts the two parallel right lines AD , EC , the outward angle BAD will be equal to the inward opposite angle AEC (I. 25.)

Q. E. D.

M 2

But

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But the angle ACE has been shewn to be equal to the angle BAD ; whence the angle ACE is also equal to the angle AEC ; and the side AE to the side AC (I. 5.)

And, since BEC is a triangle, and AD is drawn parallel to the side EC , BD will be to DC as BA is to AE (VI. 3.); and, because AE is equal to AC , BD will be to DC as BA is to AC .

Again, let BD be to DC as BA is to AC ; then will the angle BAC be bisected by the line AD .

For, let the same construction be made as before:

Then, since BD is to DC as BA is to AC , (by Hyp.), and BD to DC as BA to AE (VI. 3.), therefore, also, BA will be to AC as BA is to AE .

And since magnitudes which have the same ratio to the same magnitude are equal to each other (V. 10.), AC will be equal to AE , and the angle AEC to the angle ACE (I. 5.)

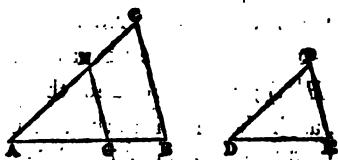
But the angle AEC is equal to the outward opposite angle BAD (I. 26.); and the angle ACE is equal to the alternate angle CAD (P. 24.): whence the angle BAD will also be equal to the angle CAD .

Q. E. D.

PROP.

PROP. V. THEOREM.

The sides about the equal angles of equiangular triangles are proportional; and if the sides about each of their angles be proportional, the triangles will be equiangular.



Let ABC , DEF be two equiangular triangles, of which BAC , EDF are corresponding angles; then will the side AB be to the side AC , as the side DE is to the side DF .

For make AG equal to DE , and AH to DF (I. 3.); and join GH :

Then, since the two sides AG , AH of the triangle AHG , are equal to the two sides DE , DF of the triangle DEF , and the angle A to the angle D , the angle AGH will also be equal to the angle DEF (I. 4.)

But the angle DEF is equal to the angle ABC (by Hyp.); consequently the angle AGH will also be equal to the angle ABC , and GH will be parallel to BC (I. 23.)

And, since the line GH is parallel to the line BC , the side AB will be to the side AC , as the side AG is to the side AH (VI. 3.)

But AG is equal to DE , and AH to DF ; whence the side AB will be to the side AC , as the side DE is to the side DF .

Q. E. D.

Again,

Again, let AB be to AC , as DE is to DF ; and AB to BC , as DE to EF ; then will the triangle ABC be equiangular with the triangle DEF .

For, let the same construction be made as before:

Then, since AB is to AC as AG is to AH (*by Hyp.*), the line GH will be parallel to the line BC (VI. 3.), and the triangle AGH will be equiangular with the triangle ABC .

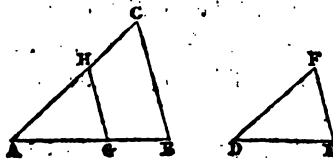
And since the sides about the equal angles of equiangular triangles are proportional (VI. 5.), the side AB will be to the side BC , as the side AG is to the side GH .

But the side AB is to the side BC , as the side DE is to the side EF (*by Hyp.*); therefore, also, the side AG will be to the side GH , as the side DE is to the side EF (V. 11.)

And, since the side AG , is equal to the side DE (*by Const.*) the side GH will also be equal to the side EF (V. 10.), and consequently the triangle DEF will be equiangular with the triangle AGH (I. 7.) or ABC , as was to be shewn.

P. R. O. P. VI. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportional, the triangles will be equiangular.



Let ABC , DEF be two triangles, having the angle A equal to the angle D , and the side AB to the side AC , as the

the side DE is to the side DF ; then will the triangle ABC be equiangular with the triangle DEF .

For, make AG equal to DE , and AH to DF ; and join GH :

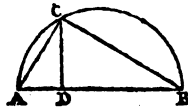
Then, since the sides AG , AH are equal to the sides DE , DF , and the angle A to the angle D (*by Hyp.*), the side GH will also be equal to the side EF , and the triangle AGH to the triangle DEF (I. 4.)

And, since AB is to AC as AG is to AH (*by Hyp.*), the line GH will be parallel to the line BC (VI. 3.); and consequently the angle AGH is equal to the angle ABC , and the angle AHG to the angle ACB (I. 25.)

The triangle ABC is, therefore, equiangular with the triangle AGH , and consequently it will also be equiangular with the triangle DEF . Q. E. D.

P R O P. VII. THEOREM.

In a right angled triangle, a perpendicular from the right angle, is a mean proportional between the segments of the hypotenuse; and each of the sides is a mean proportional between the adjacent segment and the hypotenuse.



Let ABC be a right angled triangle, and CD a perpendicular from the right angle to the hypotenuse; then will

CD

CD

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CD be a mean proportional between AD and DB; AC a mean proportional between AB and AD; and BC between AB and BD.

For, since the angle BDC is equal to the angle ACB (I. 8.), and the angle B is common, the triangles BDC, ABC will be equiangular (I. 28, Cor.)

And, in the same manner, it may be shown, that the triangles ADC, ABC are equiangular; whence the triangle ADC is also equiangular with the triangle BDC,

But the sides of equiangular triangles are proportional (VI. 5.); therefore the side AD is to the side CD, as the side CD is to the side DB,

In like manner, since the triangles ADC, BAC are each of them equiangular with the triangle ABC, AB will be to AC as AC is to AD; and AB to BC as BC is to BD (VI. 5.)

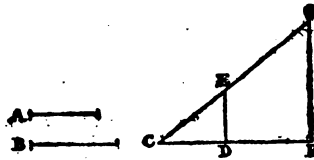
Q. E. D.

COROLL. If ACB be a semicircle, and CD a perpendicular let fall from any point C; then will $AD \times DB = DC^2$, $AB \times AD = AC^2$, and $AB \times BD = BC^2$.

PROP.

P R O P. VIII. PROBLEM.

To find a third proportional to two given right lines.



Let A, B be two given right lines; it is required to find a third proportional to them.

Draw the two indefinite right lines CF, CG , making any angle C , and take CD equal to A , and CE, DF each equal B (I. 3.)

Also join the points D, E , and make FG parallel to DE (I. 27.); and EG will be the third proportional required.

For, since CFG is a triangle, and DE is parallel to FG (by Const.) CD will be to DF as CE is to EG (VI. 3.)

But DF is equal to CE , by construction; therefore CD is to CE as CE is to EG .

And, since CD is equal to A , and CE to B (by Const.), A will be to B as B is to EG .

Q. E. D.

SCHOLIUM. A third proportional to two given right lines, may also be found by means of the last proposition.

For let AD, DC be two given right lines, one of which DC is perpendicular to the other.

Then

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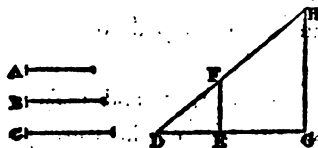
Then if CB be drawn at right angles to AC, and AD be produced till it meets the former in B, DB will be a third proportional to AD and DC as was required.

Again, let BD, DC be the two given right lines, placed at right angles to each other, as before :

Then, if CA be drawn at right angles to BC, and BD be produced till it meets the former in A, DA will be a third proportional to BD and DC.

P R O P. IX. PROBLEM.

To find a fourth proportional to three given right lines.



Let A, B, C be three given right lines ; it is required to find a fourth proportional to them.

Draw the two indefinite right lines DG, DH, making any angle D ; and take DE equal to A, DF to B, and EG to C (I. 3.)

Join the points E, F, and make GH parallel to EF (I. 27.) ; and FH will be the fourth proportional required.

For, since DCH is a triangle, and EF is parallel to GH (by Const.) DE will be to DF as EG is to FH (VI. 3.)

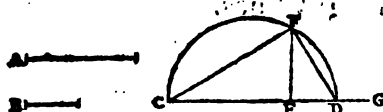
But DE is equal to A, DF to B, and EG to C (by Const.) ; therefore A will be to B as C is to FH (V. 9.)

Q. E. D.

P R O P.

PROP. X. PROBLEM.

To find a mean proportional between two given right lines.



Let A, B be two given right lines; it is required to find a mean proportional between them.

Draw the indefinite right line CG, in which take CE equal to A, and ED equal to B (I. 3.)

Upon CD describe the semi-circle CFD, and from the point E erect the perpendicular EF (I. 11.); and it will be the mean proportional required.

For join the points CF, FD:

Then, because DFC is a semi-circle, the angle CFD is a right angle (III. 16.), and consequently the triangle CDF is rectangular.

And since a perpendicular from the right angle is a mean proportional between the segments of the hypotenuse (VI. 7.), EF will be a mean proportional to CE, ED.

But CE is equal to A, and ED to B (*by Const.*); therefore EF will be a mean proportional to A and B, as was to be shewn.

SCHOLIUM. If CD, DE be the two given lines, DF will be a mean proportional to them.

And if CD, CE be the given lines, CF will be a mean proportional to them (VI. 7.)

PROP.

PROP. XI. PROBLEM.

To divide a given right line into two parts which shall have the same ratio with two given right lines.



Let CD be a given right line, and A, B two other given right lines; it is required to divide CD into two parts which shall be to each other in the ratio of A to B .

Draw the indefinite right line CG , making any angle with CD ; and make CF equal to A , and FG to B (I. 3.)

Join the points GD ; and make FE parallel to GD (I. 27.); and it will divide CD in the point E as was required.

For, since CDG is a triangle, and FE is parallel to GD (*by Conf.*), CE will be to ED as CF is to FG (VI. 3.)

But CF is equal to A , and FG to B (*by Conf.*); therefore CE will be to ED as A is to B (V. 9.)

Q. E. D.

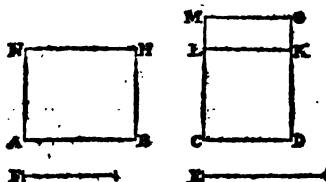
SCHOLIUM. In nearly the same manner may a third, fourth, or any other part be cut off from a given line CD .

For if CG be made the same multiple of CF that CD is of the part, and GD, FE be drawn as above, CE will be the part required.

PROP.

P R O P. XII. THEOREM.

If four right lines be proportional, the rectangle of the extremes will be equal to the rectangle of the means; and if the rectangle of the extremes be equal to the rectangle of the means, the four lines will be proportional.



Let AB be to CD as E is to F ; then will the rectangle of AB and F , be equal to the rectangle of CD and E .

For make BH perpendicular to AB , and equal to F (I. 10. 3.), and DE perpendicular to CD , and equal to E ; and in DE take DK equal to BH (I. 3.) and draw KL parallel to CD (I. 27.)

Then, since parallelograms of equal altitudes, are to each other as their bases (VI. 1.) the parallelogram $ABHG$ will be to the parallelogram $CDKE$, as AB is to CD .

And since AB is to CD as E is to F (*by Hyp.*), or as DE is to DK (*by Const.*), the parallelogram $ABHG$ will also be to the parallelogram $CDKE$ as DE is to DK (V. 11.)

But parallelograms of the same base, are to each other as their altitudes (VI. 2.); whence the parallelogram $ABHG$ will also be to the parallelogram $CDKE$ as DE is to DK .

And

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And because ratios which are the same to the same ratio are the same to each other (V. 11.), the parallelogram AH will be to the parallelogram CK, as the parallelogram CG is to the parallelogram CK.

But magnitudes which have the same ratio to the same magnitude are equal to each other (V. 10.); whence the rectangle AH is equal to the rectangle CG.

Again, let AH, the rectangle of the extremes, be equal to CG, the rectangle of the means; then will AB be to CD as E is to F.

For, let the same construction be made as before:

Then, since rectangles of equal altitudes are to each other as their bases (VI. 2.), the rectangle AH will be to the rectangle CK as AB is to CD.

And, because the rectangle AH is equal to the rectangle CG (*by Hyp.*), CG will also be to CK as AB is to CD (V. 9.)

But CG is to CK as DG is to DK or BH (VI. 2.), consequently AB will be to CD as DG is to BH. (V. 11.)

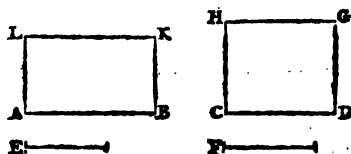
And since DG is equal to E, and BH to F, AB will be to CD as E is F (V. 9.)

Q. E. D.

PROP.

P R O P. XIII. THEOREM.

If three right lines be proportional, the rectangle of the extremes will be equal to the square of the mean : and if the rectangle of the extremes be equal to the square of the mean, the three lines will be proportional.



Let AB be to CD as CD is to E ; then will the rectangle of AB and E be equal to the square of CD.

For make BK perpendicular to AB (I. 10.) and equal to E (I. 3.) and upon CD describe the square, CG (II. 1.), and make F equal to CD.

Then, since CD is equal to F, and AB is to CD as CD is to E (*by Hyp.*), AB will also be to CD as F is to E (V. 9.)

And, since these four lines are proportional, the rectangle of AB and E will be equal to the rectangle of CD and F (VI. 12.)

But the rectangle of CD and F is equal to the square of CD, because CD is equal to F ; therefore, also, the rectangle of AB and E will be equal to the square of CD.

Again, if the rectangle of AB and E be equal to the square of CD ; AB will be to CD as CD is to E.

For let the same construction be made as before :

5

Then,

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Then, since the rectangle of AB and E is equal to the square of CD (*by Hyp.*), and the square of CD is equal to the rectangle of CD and F (II. 2.); the rectangle of AB and E will also be equal to the rectangle of CD and F.

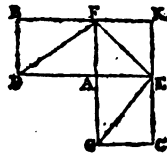
But if the rectangle of the extremes be equal to that of the means, the four lines are proportional (VI. 12.); whence AB is to CD as F is to E.

And since CD is equal to F, by construction, AB will be to CD as CD is to E.

Q. E. D.

PROP. XIV. THEOREM.

Equal parallelograms and triangles have their sides about equal angles reciprocally proportional; and if the sides about equal angles are reciprocally proportional, the parallelograms and triangles will be equal.



Let AB, AC be two equal parallelograms, and DFA AEG two equal triangles, having the angle DAF equal to the angle GAE; then will the side DA be to the side AE, as the side AG is to the side AF.

For let the sides DA, AE be placed in the same right line, and complete the parallelogram AK.

8

Then,

Then, because the angles DAF , FAE are equal to two right angles (I. 15.), and the angle FAE is equal to the angle DAG (I. 13.), the angles DAF , DAG are also equal to two right angles; and consequently FAG is a right line.

And since the parallelogram AB is equal to the parallelogram AC (*by Hyp.*); and AK is another parallelogram, AB is to AK as AC is to AK (V. 9.)

But AK is to AC as DA to AE (VI. 1.), and AK to AC as AG to AF ; whence DA is to AE as AG is to AF (V. 11.)

And, if FE be joined, it may be shewn, in like manner, that the triangle DAF is to the triangle AFE as the triangle GAE is to the triangle AFE ; and DA to AE as AG to AF .

Again, let the angle DAF be equal to the angle GAE , and the side BA to the side AE as the side AG is to the side AF ; then will the parallelogram AB be equal to the parallelogram AC , and the triangle DFA to the triangle GAE .

For since DA is to AE as AG to AF (*by Hyp.*), and DA to AE as AB to AK (VI. 1.), AG will be to AF as AB to AK (V. 11.)

But AG is to AF as AC to AK (VI. 1.); whence AB is to AK as AC to AK (V. 11.); and consequently the parallelogram AB is equal to the parallelogram AC (V. 10.)

And since triangles are the halves of parallelograms, which have the same base and altitude, the triangle DFA will be equal to the triangle GAE .

Q. E. D.

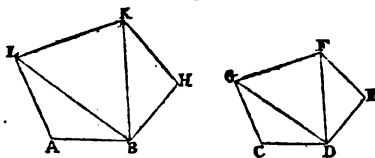
COROLL. The sides of equal rectangles are reciprocally proportional; and if the sides are reciprocally proportional, the rectangles will be equal.

N

PROP.

PROP. XV. PROBLEM.

Upon a given right line to describe a rectilineal figure, similar, and similarly situated, to a given rectilineal figure.



Let AB be the given right line, and $CDEFG$ the given rectilineal figure; it is required to describe a rectilineal figure upon AB , which shall be similar, and similarly situated to $CDEFG$.

Join DC , DF ; and at the points A , B , make the angles BAL , ABL equal to the angles DCG , CDG (I. 20.)

In like manner, at the points B , L , make the angles BLK , LBK equal to the angles DGF , GDF : and BKH , KBH equal to DFE , FDE .

Then, because two angles in one triangle are equal to two angles in another, each to each, the remaining angles in each of the corresponding triangles, will also be equal (I. 28. *Cor.*)

And since the angles ALB , BLK , are equal to the angles CGD , DGF , and LKB , BKH to GFD , DFE , the angle ALK will be equal to the angle CGF , and the angle LKH to the angle GFE .

And, in the same manner, it may be shewn that the angles KHB , HBA and BAL are equal to the angles FED , EDC and DCG .

The

The figures $ABHKL$ and $CDEFG$ are, therefore, equiangular: and they have their sides about the equal angles, also, proportioned.

For, since the triangles ALB , CGD are equiangular, AL will be to LB as CG to GD (VI. 5.); or AL to CG as LB to GD (V. 15.)

And, in like manner, LK will be to LB as GF to GD (VI. 5.); or LK to GF as LB to GD (V. 15.)

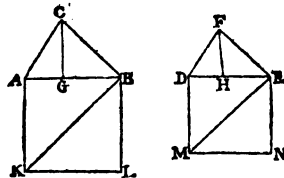
But ratios which are the same to the same ratio are the same to each other (V. 11.); whence AL will be to CG as LK to GF ; or AL to LK as CG to GF (V. 15.)

And, in the same manner, it may be shewn, that the sides about the angles K , H , B , A , are proportional to the sides about the angles F , E , D , C .

The figure $ABHKL$ is, therefore, similar, and similarly situated with the figure $CDEFG$ (VI. Def. 1.); and it is described upon the right line AB ; as was to be done.

PROP. XVI. THEOREM.

Equiangular, or similar triangles, are to each other as the squares of their homologous sides.



Let ABC , DEF be two similar triangles, of which the sides AB , DE are homologous; then will the triangle ABC

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be to the triangle DEF as the square of AB is to the square of DE.

For, on AB, DE describe the squares AL, DN (II. 1.), and let fall the perpendiculars CG, FH (I. 12.)

Then, since the triangles ABC, DEF are similar (*by Hyp.*), AC will be to AB as DF to DE (VI. Def. 1.); or AC to DF as AB to DE (V. 15.)

And, because the triangles AGC, DHF are equiangular, AC will be to CG as DF to FH (VI. 5.); or AC to DF as CG to FH (V. 15.)

But ratios which are the same to the same ratio, are the same to each other (V. 11.); therefore CG is to FH as AB to DE; or CG to AB as FH to DE (V. 15.)

And since triangles which have the same base, are to each other as their altitudes (VI. 2.), the triangle ABC is to the triangle AKB as CG is to AK, or AB.

In the same manner it may be shewn, that the triangle DEF is to the triangle DME as FH is to DM, or DE.

But CG has been shewn to be to AB as FH is to DE; therefore the triangle ABC is to the triangle AKB as the triangle DEF is to the triangle DME (V. 11.)

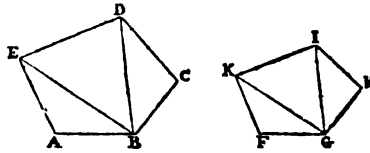
And since the square AL is double the triangle AKB (I. 32.), and the square DN is double the triangle DME, the triangle ABC will be to the triangle DEF as the square AL is to the square DN (V. 12 and 15.)

Q. E. D.

PROP.

P R O P. XVII. THEOREM.

Similar polygons are to each other as the squares of their homologous sides.



Let $ABCDE$, $FGHIK$ be similar polygons, of which AB , FG are homologous sides ; then will the polygon $ABCDE$ be to the polygon $FGHIK$ as the square of AB is to the square of FG .

For join the points BE , BD , GK and GI :

Then, since the angle A is equal to the angle F , and AB is to AE as FG is to FK (VI. *Def.* 1.), the triangles EAB , KFG will be equiangular, or similar (VI. 6.)

And if, from the equal angles AED , FKI , there be taken the equal angles AEB , FKG , the remaining angles BED , GKI will also be equal to each other.

But ED is to KI as EA is to KF (VI. *Def.* 1. and V. 15.), and EA is to KF as EB to KG (VI. 5. and V. 15.); whence ED will be to KI as EB is to KG (V. 11.)

Since, therefore, the angles BED , GKI are equal to each other, and the sides about them are proportional, the triangles BED , GKI will, also, be equiangular, or similar (VI. 6.)

And, in the same manner, it may be shewn, that the triangles BCD , GHI are equiangular, or similar.

N 3

But

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AE, AF; whence AB is to AC as the rectangle of AD, AF is to the rectangle of AG, AE (V. 11 and 15.)

Again, let DFA, AEG be two triangles, having the angle DAF equal to the angle GAE, then will DFA be to AEG as the rectangle of DA, AF is to the rectangle of GA, AE.

For let the sides DA, AE be placed in the same right line; and complete the parallelograms AB, AC, AK.

Then, as before, AB is to AC as the rectangle of DA, AF is to the rectangle of AG, AE.

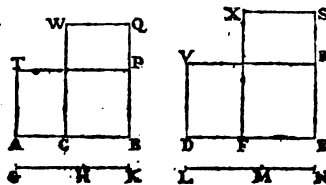
But the triangles DFA, AEG are half the parallelograms AB, AC (I. 30.); whence DFA is to AEG as the rectangle of DA, AF is to the rectangle of GA, AE.

Q. E. D.

SCHOLIUM. If the line EF be drawn, the latter part of this proposition may be proved from the triangles, independently of the former,

P R O P. XIX. THEOREM.

The rectangles under the corresponding lines of two ranks of proportionals, are themselves proportionals.



Let AB be to CB as DE is to FE, and GH to GK as LM to LN; then will the rectangle of AB, GH be to that of CB,

CB, GK as the rectangle of DE, LM is to that of FE, LN.

For draw BQ, ES at right angles to AB, DE (I. 11.), and make BP equal to GH, BQ to GK, ER to LM, and ES to LN (I. 3.): and complete the parallelograms AP, CQ, DR and FS.

Then since parallelograms, of the same altitude, are to each other as their bases (VI. 1.), AP will be to CP as AB to CB; and DR to FR as DE to FE.

But AB is to CB as DE to FE (*by Hyp.*); whence AP will be to CP as DR to FR (V. 11.), or AP to DR as CP to FR (V. 15.)

And since parallelograms of the same base are to each other as their altitudes (VI. 2.), CP will be to CQ as BP to BQ; and FR to FS as ER to ES.

Or, because BP, BQ are equal to GH, GK, and ER, ES to LM, LN (*by Const.*), CP will be to CQ as GH to GK; and FR to FS as LM to LN (V. 9.)

But GH is to GK as LM to LN (*by Hyp.*), therefore CP will be to CQ as FR is to FS (V. 11.), or CP to FR as CQ to FS (V. 15.)

And it has been before shewn that AP is to DR as CP to FR; whence AP is to DR as CQ to FS (V. 11.), or AP to CQ as DR to FS (V. 15.)

But AP is the rectangle of AB, GH; CQ of CB, GK; DR of DE, LM; and FS of FE, LN (*by Const.*); therefore the rectangle of AB, GH is to that of CB, GK as the rectangle of DE, LM is to that of FE, LN.

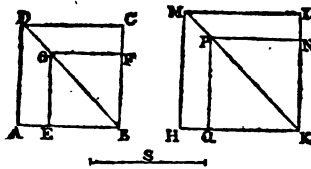
Q. E. D.

COROLL. The squares of four proportional lines are themselves proportionals.

PROP.

P R O P. XX. THEOREM.

The sides and diagonals of four proportional squares, are themselves proportional.



Let AC, EF, HL and QN be four proportional squares; then will their sides and diagonals be also proportionals.

For, make s a fourth proportional to AB, DB and HK (VI. 9.); and draw the diagonals BD and KM.

Then, since AB is to EB as HK is to s (*by Const.*), AC will be to EF as HL is to the square of s (VI. 19. *Cor.*)

And because AC is to EF as HL is to QN (*by Hyp.*), HL will be to the square of s as HL is to QN, or the square of QK (V. 11.)

But magnitudes which have the same ratio to the same magnitude are equal to each other (V. 10.); whence the square of s is equal to the square of QK.

And since equal squares have equal sides (II. 3.), s is equal to QK; and consequently AB is to EB as HK to QK.

Again, because the triangles ABD, EBG are equiangular, AB will be to EB as BD to BG (VI. 5.)

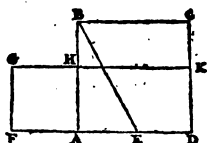
And because the triangles HKM, QKP are also equiangular, HK will be to QK, as KM to KP (VI. 5.)

But AB has been shewn to be to EB as HK is to QK; consequently BD is to BG as KM is to KP (V. 11.)

Q. E. D.
PROP.

P. R O P. XXI: PROBLEM.

To cut a given right line in extreme and mean proportion.



Let AB be the given right line; it is required to cut it in extreme and mean proportion.

Upon AB describe the square AC (II. 1.), and bisect the side AD in E (I. 10.) and join BE.

In EA produced, take EF equal to EB (I. 3.); and upon AF describe the square FH (II. 1.); then will AB be divided at the point H as was required.

For since DF is the sum of EB, ED, or EB, EA, and AF is their difference, the rectangle of DF, FA is equal to the difference of the squares of EB, EA (II. 13.)

But the rectangle of DF, FA is equal to DG, because FA is equal to FC; and the difference of the squares of EB, EA is equal to the square of AB (II. 14. Cor.); whence DG is equal to AC.

And if from each of these equals, the part AK, which is common, be taken away, the remainder AG will be equal to the remainder HC.

But equal parallelograms have the sides about equal angles reciprocally proportional (VI. 15.); whence HK is to HG as HA to HB.

And since HK is equal to AD, or AB, and HG to HA, AB will be to HA as HA is to HB.

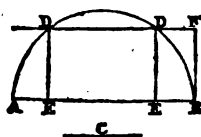
Q. E. D.

PROP.

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P R O P. XXII. PROBLEM.

To divide a given right line into two such parts, that their rectangle may be equal to a given square, the side of which is not greater than half the given line.



Let AB be the given line, and c the side of the given square; it is required to divide AB into two such parts that their rectangle may be equal to the square of c .

Upon AB describe the semicircle BDA , and make BF perpendicular to AB (I. 11.), and equal to c . (I. 3.)

Through F draw FD parallel to AB (I. 27.); and from the point D where it cuts the circle, let fall the perpendicular DE (I. 12.); and AB will be divided at E as was required.

For since BDA is a semicircle (*by Const.*), and DE is perpendicular to the diameter AB (*by Const.*), the rectangle of AE , EB will be equal to the square of ED (VI. 7. *Cor.*)

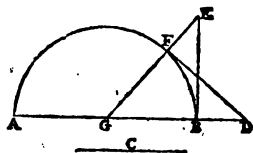
But ED is equal to FB (I. 30.) or c ; whence the rectangle of AE , EB will be equal to the square of c as was to be shewn.

SCHOLIUM. When BF , or c , is equal to half AB , FD will be a tangent to the circle, and the rectangle of AE , EB will be the greatest possible.

PROP.

P R O P. XXIII. PROBLEM.

To a given right line to add another right line such, that the rectangle of the whole and the part added shall be equal to a given square.



Let AB be the given line, and c the side of the given square; it is required to add a line to AB such, that the rectangle of the whole and the part added shall be equal to the square of c .

Make BE perpendicular to AB (I. 11.), and equal to c (I. 3.); also bisect AB in G (I. 10.), and join GE .

Then, if AB be produced, and GD be taken equal to GE , (I. 3.), the part BD will be added to AB , as was required.

For on AB describe the semicircle BFA , cutting GE in F , and join FD .

Then, since the two sides GB , GE of the triangle GEB , are equal to the two sides GF , GD , of the triangle GDF , and the angle G is common, the angle GBE will be equal to the angle GFD , and the side FD to the side BE (I. 4.)

But the angle GBE is a right angle (*by Constr.*); whence the angle GFD is also a right angle; and consequently FD is a tangent to the circle at F (III. 10.)

And

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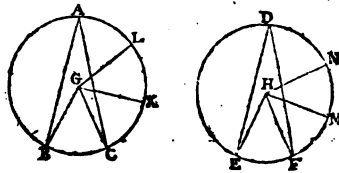
And since DF is a tangent to the circle, and DA is drawn to the opposite part of the circumference, the rectangle of AD , DB will be equal to the square of DF (III. 29.)

But DF has been shewn to be equal to BE or c ; whence the rectangle of AD , DB will also be equal to the square of c .

Q. E. D.

PROP. XXIV. THEOREM.

Angles at the centres or circumferences of equal circles, have the same ratio with the arcs on which they stand.



Let ABC , DEF be two equal circles, in which BGC , EHF are angles at the centre, and BAC , EDF angles at the circumference; then will the arc BC be to the arc EF as the angle BGC is to the angle EHF , or as the angle BAC to the angle EDF .

For on the circumference of the circle ABC take any number of arcs whatever CK , KL each equal to BC ; and on the circumference of the circle DEF any number of arcs whatever FM , MN , each equal to EF ; and join GK , GL , HM , HN .

Then, because the arcs BC , CK , KL are all equal to each other, the angles BGC , CGK , KGL will also be equal to each other (III. 21.)

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And,

And, therefore, whatever multiple the arc BL is of the arc BC , the same multiple will the angle BGL be of the angle BGC .

For the same reason, whatever multiple the arc EN is of the arc EF , the same multiple will the angle ENH be of the angle EHF .

If, therefore, the arc BL be equal to the arc EN , the angle BGL will be equal to the angle ENH ; and if equal, equal; and if less, less.

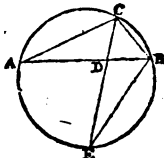
But BL and BGL are any equimultiples whatever of BC and BGC , and EN and ENH of EF and EHF ; whence the arc BC is to the arc EF as the angle BGC is to the angle EHF (V. 5.)

And since the angle BGC is double the angle BAC , and the angle EHF is double the angle EDF (III. 14.), the arc BC will also be to the arc EF as the angle BAC is to the angle EDF (V. 13.)

Q. E. D.

P R O P. XXV. THEOREM.

The rectangle of the two sides of any triangle, is equal to the rectangle of the segments of the base, made by a line bisecting the vertical angle, together with the square of that line.



Let ABC be a triangle, having the angle ACB bisected by the right line CD ; then will the rectangle of AC , CB

be

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be equal to the rectangle of AD, DB, together with the square of CD.

For, about the triangle ABC, describe the circle AEC (IV. 5.), cutting CD, produced, in E; and join EB.

Then, because the angle ACD is equal to the angle ECB (*by Hyp.*), and the angle CAD to the angle CEB (III. 15.), the remaining angle ADC will be equal to the remaining angle CBE (I. 28. *Cqr.*)

The triangles CAD, CEB being, therefore, equiangular, CA will be to CD as CE to CB (VI. 5.); and consequently the rectangle of CA, CB is equal to the rectangle of CE, CD (VI. 12.)

But the rectangle of CE, CD is equal to the rectangle of ED, DC, together with the square of CD (II. 10.); whence the rectangle of CA, CB is also equal to the rectangle of ED, DC, together with the square of CD.

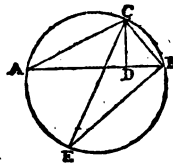
And since the rectangle of ED, DC is equal to the rectangle of AD, DB (III. 27.), the rectangle of AC, CB is also equal to the rectangle of AD, DB, together with the square of CD.

Q. E. D.

PROP.

P R O P. XXVI. THEOREM.

The rectangle of the two sides of any triangle, is equal to the rectangle of the perpendicular, drawn from the vertical angle to the base, and the diameter of the circumscribing circle.



Let ABC be a triangle, having CD perpendicular to AB ; then will the rectangle of AC , CB be equal to the rectangle of CD and the diameter of the circumscribing circle.

For, about the triangle ABC , describe the circle AEC (IV. 5.); in which draw the diameter CE ; and join EB .

Then, since the angle CAD is equal to the angle CEB (III. 15.) and the angle ADC to the angle EBC , being each of them right angles (*Const. and* III. 16.), the remaining angle ACD will be equal to the remaining angle ECB (I. 28. *Cor.*)

The triangles ACD , ECB are, therefore, equiangular; whence AC is to CD as CE is to CB (VI. 5.); and consequently the rectangle of AC , CB is equal to the rectangle of CD , CE (VI. 12.)

Q. E. D.

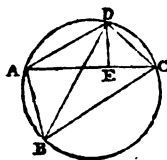
SCHOLIUM. When ABC is an obtuse angle, the perpendicular CD falls without the circle; but the same demonstration will hold.

○

P R O P.

P R O P. XXVII. THEOREM.

The rectangle of the two diagonals of any quadrilateral, inscribed in a circle, is equal to the sum of the rectangles of its opposite sides.



Let $ABCD$ be any quadrilateral inscribed in a circle, of which the diagonals are AC, BD ; then will the rectangle of AC, BD be equal to the rectangles of AB, DC and AD, BC .

For make the angle CDE equal to the angle ADB (I. 20.); then, if to each of these angles, there be added the common angle EDB , the angle ADE will be equal to the angle CDB .

The angle DAE is also equal to the angle DBC , being angles in the same segment, whence the remaining angle AED is equal to the remaining angle BCD (I. 28. *Cor.*) \circ

Since, therefore, the triangles ADE, BDC are equiangular, AD is to AE as BD is to BC (VI. 5.); and consequently the rectangle of AD, BC is equal to the rectangle of AE, BD (VI. 12.)

Again, the angle CDE being equal to the angle ADB (*by Constr.*), and the angle ECD to the angle ABD (III. 15.), the remaining angle CED will be equal to the remaining angle BAD (I. 28. *Cor.*)

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The triangles CED , ADB are, therefore, also equiangular; whence AB is to BD as EC is to DC (VI. 5.); and consequently the rectangle of AB , DC is equal to the rectangle of EC , BD . (VI. 12.)

And if, to these equals, there be added the former, the rectangle of AB , DC together with the rectangle of AD , BC will be equal to the rectangle of EC , BD together with the rectangle of AE , BD .

But the rectangles of AE , BD , and EC , BD are equal to the rectangle of AC , BD (II. 8.); whence the rectangle of AC , BD is also equal to the rectangles of AB , DC and AD , BC .

Q. E. D.

BOOK VII.

DEFINITIONS.

1. The common section of two planes, is the line in which they meet, or cut each other.

2. A right line is perpendicular to a plane, when it is perpendicular to every right line which meets it in that plane.

3. A plane is perpendicular to a plane, when every right line in the one, which is perpendicular to their common section, is perpendicular to the other.

4. The inclination of a right line to a plane, is the angle it makes with another line, drawn from the point of section, to that point in the plane, which is cut by a perpendicular falling from any part of the former.

5. The inclination of a plane to a plane, is the angle contained by two right lines, drawn from any point in the common section, at right angles to that section; one in one plane, and the other in the other.

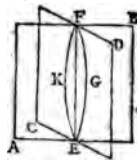
6. Parallel planes, are such as being produced ever so far both ways will never meet.

7. A plane is said to be extended by, or to pass through a right line, when every part of that line lies in the plane.

PROP.

P R O P. I. THEOREM.

The common section of any two planes is a right line.



Let AB, CD be two planes, whose common section is EF ; then will EF be a right line.

For if not, let FGE be a right line, drawn in the plane AB ; and FKE another right line, drawn in the plane CD .

Then, since the lines FGE, FKE are in different planes, they must fall wholly without each other.

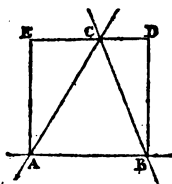
But the line FGE , having the same extremities with the line FKE , will coincide with it: whence they coincide and fall wholly without each other, at the same time, which is absurd.

The lines FGE, FKE cannot, therefore, be right lines; and consequently the line EF , which lies in each of the planes, must be a right line, as was to be shewn.

SCHOLIUM. One part of a right line cannot be in a plane, and another part out of it. For since the line can be produced in that plane, the part out of the plane, and the part produced would have different directions, which is absurd.

P R O P. II. THEOREM.

Any three right lines which mutually intersect each other, are all in the same plane.



Let AB , BC , CA be three right lines, which intersect each other in the points A , B , C ; then will those lines be in the same plane.

For let any plane AD pass through the points A , B , and be turned round that line, as an axis, till it pass through the point C .

Then, because the points A , C are in the plane AD , the whole line AC must also be in it; or otherwise its parts would not lie in the same direction.

And, because the points B , C are also in this plane, the whole line BC must likewise be in it; for the same reason.

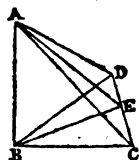
But the line AB is in the plane AD , by hypothesis; whence the three lines AB , BC , CA are all in the same plane, as was to be shewn.

COR. Any two right lines which intersect each other, are both in the same plane; and through any three points a plane may be extended,

P R O P.

PROP. III. THEOREM.

If a right line be perpendicular to two other right lines, at their point of intersection, it will also be perpendicular to the plane which passes through those lines.



Let the right line AB be perpendicular to each of the two right lines BC, BD, at their point of intersection B; then will it also be perpendicular to the plane which passes through those lines.

For make BD equal to BC; and, in the plane which passes through those lines, draw any right line BE; and join the points CD, AD, AE and AC:

Then because the side BC is equal to the side BD (*by Const.*), and the perpendicular AB is common to each of the triangles ABC, ABD, the side AC will also be equal to the side AD (I. 4.)

And since the triangles CAD, CBD are isosceles, the rectangle of CE, ED, together with the square of EB, is equal to the square of DB; and the rectangle of CE, ED together with the square of EA, is equal to the square of AD (II. 20.)

From each of these equals, take away the rectangle of CE, ED which is common, and the difference of the

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Squares of EB , EA will be equal to the difference of the squares of DB , AD .

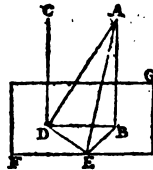
But the difference of the squares of DB , AD is equal to the square of AB (II. 14 *Cor.*); whence the difference of the squares of EB , EA will also be equal to the square of AB ; and consequently AB is perpendicular to BE , as was to be shewn.

COROLL. If a right line be perpendicular to three other right lines, at their point of intersection, those lines will be all in the same plane.

For if either of them, as BE , were above or below the plane which passes through the other two, the angle ABE would be less or greater than a right angle.

P R O P. IV. THEOREM.

If two right lines be perpendicular to the same plane, they will be parallel to each other.



Let the right lines AB , CD be each of them perpendicular to the plane FG , then will those lines be parallel to each other.

For join the points D , B ; and, in the plane FG , make DE perpendicular to DB , and equal to AB (I. 11. 3.); and join the points AE , AD .

∴

Then,

Then, since the right lines AB, CD are perpendicular to the plane FG (*by Hyp.*), the angles ABD, ABE, CDB and CDE will be right angles (VII. Def. 2.)

And because the side AB, is equal to the side ED (*by Conf.*), the side DB common to each of the triangles BAD, BED, and the angles ABD, BDE right angles (*by Hyp. and Conf.*), the side AD will also be equal to the side EB (I. 4.)

Again, since the sides AD, DE are equal to the sides EB, BA, and the side AE is common to each of the triangles EBA, EDA, the angle ADE will also be equal to the angle ABE (I. 7.), and is, therefore, a right angle.

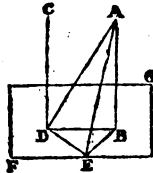
And, because the line ED is at right angles with each of the three lines DA, DB, DC, those lines, together with the line AB, will be all in the same plane (VII. 3. Cor.)

Since, therefore, the lines AB, BD, DC are all in the same plane, and the angles ABD, CDB are each of them a right angle, the line AB will be parallel to the line CD (I. 23.), as was to be shewn.

COR. Any two parallel right lines AB, CD, are in the same plane; and any right line DA, which intersects those parallels, is in the same plane with them.

PROP. V. THEOREM.

If two right lines be parallel to each other, and one of them be perpendicular to a plane; the other will also be perpendicular to that plane.



Let AB , CD be two parallel right lines, one of which, AB , is perpendicular to the plane FG ; then will the other CD be also perpendicular to that plane.

For join the points D , B ; and, in the plane FG , make DE perpendicular to DB , and equal to BA (I. 11. 3.); and join AE , AD and ER :

Then, because AB is perpendicular to the plane FG (*by Hyp.*) the angles ABD , ABE will be right angles (VII. Def. 2.)

And, since the side AB is equal to the side ED (*by Hyp.*), the side DB common to each of the triangles BAD , BED , and the angles ABD , BDE right angles (*by Const. and Hyp.*), the side AD will be equal to the side EB (I. 4.)

Again, since the sides AD , DE are equal to the sides EB , BA , and the side AE is common to each of the triangles EAD , EBD , the angle ADE will be equal to the angle ABE (I. 7.), and is, therefore, a right angle.

And, because the right lines AB , CD are parallel to each other (*by Hyp.*), and the line AD intersects them, they will be all in the same plane (VII. 4. Cor.); and the angle

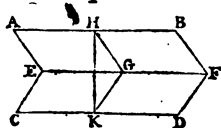
angle ABD being a right angle, the angle CDB will also be a right angle (I. 25.)

But since ED is at right angles to DB, DA, it is also at right angles to the plane which passes through them (VII. 3.); and consequently to DC (VII. Def. 2.)

The line DC is, therefore, perpendicular to each of the lines DE, DB; whence it is also perpendicular to the plane EC (VII. 3.), as was to be shewn.

P R O P. VI. THEOREM.

If two right lines be parallel to the same right line, though not in the same plane with it, they will be parallel to each other.



Let the right lines AB, CD be each of them parallel to the right line EF, though not in the same plane with it; then will AB be parallel to CD.

For take any point G in the line EF, and draw the right lines GH, GK, each perpendicular to EF (I. 11.), in the planes AF, ED of the proposed parallels:

Then since the right line EF is perpendicular to the two right lines GH, GK, at their point of intersection G, it will also be perpendicular to the plane HGK which passes through those lines (VIII. 3.)

And because the lines AB, EF are parallel to each other (by Hyp.), and one of them, EF, is perpendicular to the plane HGK, the other, AB, will also be perpendicular to that plane (VII. 5.)

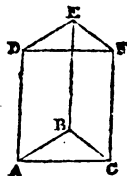
And,

And, in like manner, it may be proved that the line CD is also perpendicular to the plane HGK .

But when two right lines are perpendicular to the same plane, they are parallel to each other (VII. 4.); whence the line AB is parallel to the line CD , as was to be shewn.

P R O P. VII. THEOREM.

If two right lines that meet each other, be parallel to two other right lines that meet each other, though not in the same plane with them, the angles contained by those lines will be equal.



Let the two right lines AB , BC , which meet each other in the point B , be parallel to the two right lines DE , EF which meet each other in the point E ; then will the angle ABC be equal to the angle DEF .

For make BA , BC , ED , EF all equal to each other (I. 3.), and join AD , CF , BE , AC and DF .

Then, because BA is equal and parallel to ED (*by Hyp.*), AD will be equal and parallel to BE (I. 29.)

And, for the same reason, CF will also be equal and parallel to BE .

But lines which are equal and parallel to the same line, though not in the same plane with it, are equal and parallel

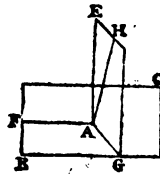
parallel to each other (I. 26. and VII. 6.); whence AD is equal and parallel to CF.

And since lines which join the corresponding extremes of two equal and parallel lines are also equal and parallel (I. 29.), AC will be equal and parallel to DF.

The three sides of the triangle ABC are, therefore, equal to the three sides of the triangle DEF, each to each; whence the angle ABC is equal to the angle DEF (I. 7.), as was to be shewn.

P R O P. VIII. PROBLEM.

To draw a right line perpendicular to a given plane, from a given point in the plane.



Let A be the given point, and BC the given plane; it is required to draw a right line from the point A that shall be perpendicular to the plane BC.

Take any point E above the plane BC, and join EA; and through A draw AF, in the plane BC, at right angles with EA (I. 11.); then if EA be also at right angles with any other line which meets it in that plane; the thing required is done.

But if not, in the plane BC, draw AG at right angles to AF (I. 11.); and in the plane EG, which passes through the points E, A, G, make AH perpendicular to

AG

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AG (I. 11.), and it will also be perpendicular to the plane BC, as was required.

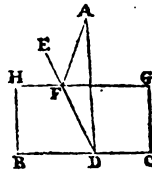
For, since the right line FA is perpendicular to each of the right lines AE, AG (*by Const.*), it will also be perpendicular to the plane EG which passes through those lines (VII. 3.)

And because a right line which is perpendicular to a plane is perpendicular to every right line which meets it in that plane (*Def. 2.*), FA will be perpendicular to AH.

But AG is also perpendicular to AH (*by Const.*); whence AH, being perpendicular to each of the right lines FA, AG, it will also be perpendicular to the plane BC (VII. 3.), as was to be shewn.

P R O P. IX. PROBLEM.

To draw a right line perpendicular to a given plane, from a given point above it.



Let A be the given point, and BC the given plane; it is required to draw a right line from the point A that shall be perpendicular to the plane BC.

Take any right line BC, in the plane BC, and draw AD perpendicular to BC (I. 11.); then if it be also perpendicular to the plane BC, the thing required is done.

But if not, draw DE, in the plane BC, at right angles to BC (I. 11.), and make AF perpendicular to DE (I. 12.); then

then will AF be perpendicular to the plane BC , as was required.

For, through the point F , draw the line HG parallel to the line BC (I. 27.)

Then since the right line BC is perpendicular to each of the right lines DA , DE , it will also be perpendicular to the plane which passes through those lines (VII. 3.)

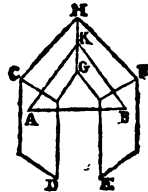
And because the lines BC , HG are parallel to each other, and one of them, BC , is perpendicular to the plane ADF , the other, HG , will also be perpendicular to that plane (VII. 5.)

But if a line be perpendicular to a plane it will be perpendicular to all the lines which meet it in that plane (VII. Def. 2.); whence the line HG is perpendicular to AF .

And since the line AF is perpendicular to each of the lines HG , ED , at their point of intersection F , it will also be perpendicular to the plane BC (VII. 3.), as was to be shewn.

P R O P. X. THEOREM.

Planes to which the same right line is perpendicular, are parallel to each other.



Let the right line AB be perpendicular to each of the planes CD , EF ; then will those planes be parallel to each other.

For

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For if they be not, let them be produced till they meet each other ; and in the line GH , which is their common section, take any point K ; and join KA , KB :

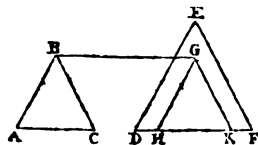
Then, because the line AB is perpendicular to the plane EF (*by Hyp.*), it will also be perpendicular to the line BK , which lies in that plane (*VII. Def. 2.*) ; and the angle ABK will be a right angle.

And, for the same reason, the line AB , which is perpendicular to the plane DC (*by Hyp.*), will be perpendicular to the line AK ; and the angle BAK will also be a right angle.

The angles ABK , BAK are, therefore, equal to two right angles, which is absurd (*I. 28.*) ; and consequently the planes can never meet, but must be parallel to each other (*VII. Def. 6.*), as was to be shewn.

P R O P. XI. THEOREM.

If two right lines which meet each other, be parallel to two other right lines which meet each other, though not in the same plane with them, the planes which pass through those lines will be parallel.



Let the right lines AB , BC , which meet each other in B , be parallel to the right lines DE , EF , which meet each other in E , though not in the same plane with them ; then will the plane ABC be parallel to the plane DEF .

For

For through the point B draw BG perpendicular to the plane DFE (VII. 9.); and make GH parallel to DE, and GK to EF (I. 27.)

Then because BG is at right angles with the plane DFE, it will also be at right angles with each of the lines GH, GK which meet it in that plane (*Def. 2.*)

And since GH is parallel to DE or AB (*by Constr. and VII. 6.*), and BG intersects them, the angles BGH, GBA are, together, equal to two right angles (I. 25.)

But the angle BGH has been shewn to be a right angle; whence the angle GBA is also a right angle; and consequently GB is perpendicular to BA.

And, in the same manner, it may be shewn, that GB is perpendicular to BC.

The right line GB, therefore, being perpendicular to each of the right lines BA, BC, will also be perpendicular to the plane ACB through which they pass (VII. 3.)

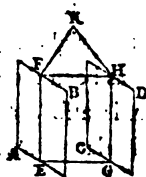
But planes to which the same right line is perpendicular are parallel to each other (VII. 10.); whence the plane ACB is parallel to the plane DFE.

Q. E. D.



P R O P. XII. THEOREM.

If any two parallel planes be cut by another plane, their common sections will be parallel.



Let the two parallel planes AB , CD be cut by the plane $EFGH$; then will their common sections EF , GH be parallel to each other.

For if EF , GH be not parallel, they may be produced till they meet, either on the side FH , or the side EG .

Let them be produced on the side FH , and meet each other in the point K .

Then, since the whole line EFK is in the plane AB , or the plane produced, the point K must be in that plane.

And because the whole line GHK is in the plane CD , or the plane produced, the point K must also be in that plane.

Since, therefore, the point K is in each of the planes AB , CD , those planes, if produced, will meet in that point.

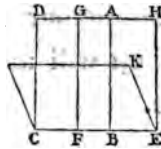
But the two planes are parallel to each other, by hypothesis; whence they meet, and are parallel, at the same time, which is absurd.

The

The lines EF , GH , therefore, do not meet on the side FH ; and, in the same manner, it may be proved, that they do not meet on the side EG ; consequently they are parallel to each other. Q. E. D.

P R O P. XIII. THEOREM.

If a right line be perpendicular to a plane, every plane which passes through it will also be perpendicular to that plane.



Let the right line AB be perpendicular to the plane CK ; then will every plane which passes through that line be also perpendicular to CK .

For let ED be any plane which passes by the line AB ; and in this plane draw any right line GF perpendicular to the common section CE (I. 11.)

Then, because the line AB is perpendicular to the plane CK (*by Hyp.*), it will also be perpendicular to the line CE ; and the angle ABF will be a right angle (VII. Def. 2.)

And since the angles ABF , GFB are each of them a right angle, and the lines AB , GF are in the same plane, they will be parallel to each other (VII. 4.)

Since, therefore, these lines are parallel to each other, and one of them, AB , is perpendicular to the plane CK ,

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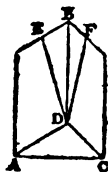
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the other, GF , will also be perpendicular to that plane (VII. 5.)

But one plane is perpendicular to another, when any right line that can be drawn in it, at right angles to the common section, is also at right angles to the other plane (VII. Def. 3.); whence the plane ED is perpendicular to the plane CK , as was to be shewn.

P R O P. XIV. THEOREM.

If two planes which cut each other, be each of them perpendicular to a third plane, their common section will also be perpendicular to that plane.



Let the two planes AB , CB be each of them perpendicular to the plane ACD ; then will their common section BD be also perpendicular to ACD .

For if not, let DE be drawn in the plane AB , at right angles to the common section AD ; and DF in the plane CB at right angles to the common section DC (I. 11.)

Then because the plane AB is perpendicular to the plane ACD (*by Hyp.*), the line DE will also be perpendicular to that plane (VII. 3.)

And

And since the plane CB is perpendicular to the plane ACD (*by Hyp.*), the line DF will also be perpendicular to that plane (VII. 3.)

But lines which are perpendicular to the same plane are parallel to each other (VII. 4.); whence the lines DE , DF meet, and are parallel at the same time, which is absurd.

These lines, therefore, are not perpendicular to the plane ACD ; and the same may be shewn of any other line but DB ; whence DB is perpendicular to ACD , as was to be shewn.

BOOK VIII.

DEFINITIONS.

1. A solid angle is that which is made by three or more plane angles, which meet each other in the same point.
2. Similar solids, contained by plane figures, are such as have all their solid angles equal, each to each, and are bounded by the same number of similar planes.
3. A prism is a solid whose ends are parallel, equal, and like plane figures, and its sides parallelograms.
4. A parallelepiped is a prism contained by six parallelograms, every opposite two of which are equal, alike, and parallel.
5. A rectangular parallelepiped is that whose bounding planes are all rectangles, which are perpendicular to each other.
6. A cube is a prism, contained by six equal square sides, or faces.
7. A pyramid is a solid whose base is any right lined plane figure, and its side triangles, which meet each other in a point above the base, called the vertex.
8. A cylinder is a solid generated by the revolution of a right line about the circumferences of two equal and parallel circles, which remain fixed.
9. The axis of a cylinder is the right line joining the centres of the two parallel circles, about which the figure is described.

10. A

10. A cone is a solid generated by the revolution of a right line about the circumference of a circle, one end of which is fixed at a point above the plane of that circle.

11. The axis of a cone is the right line joining the vertex, or fixed point, and the centre of the circle about which the figure is described.

12. Similar cones and cylinders are such as have their altitudes and the diameters of their bases proportional.

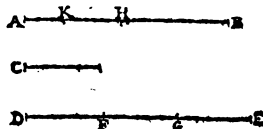
13. A sphere is a solid generated by the revolution of a semi-circle about its diameter, which remains fixed.

14. The axis of a sphere is the right line about which the semi-circle revolves; and the centre is the same as that of the semi-circle.

15. The diameter of a sphere is any right line passing through the centre, and terminated both ways by the surface.

P R O P. I. LEMMA.

If from the greater of two magnitudes, there be taken more than its half; and from the remainder, more than its half; and so on: there will at length remain a magnitude less than the least of the proposed magnitudes.



Let AB and C be any two magnitudes, of which AB is the greater; then, if from AB there be taken more than

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its half; and from the remainder more than its half; and so on: there will at length remain a magnitude less than c .

For since AB and c are each finite magnitudes, it is evident that c may be taken such a number of times as at length to become greater than AB .

Let, therefore, DE be such a multiple of c as is greater than AB , and divide it into the parts DF , FG , GE , each equal to c .

Also from AB take BH greater than its half; and from the remainder AH , take HK greater than its half, and so on, till there be as many divisions in AB as there are in DE .

Then because DE is greater than AB ; and BH , taken from AB , is greater than its half, but EG , taken from DE , is not greater than its half; the remainder GD will be greater than the remainder HA .

And, again, because GD is greater than HA , and HK , taken from HA , is greater than its half, but GF , taken from GD , is not greater than its half; the remainder FD will be greater than the remainder AK .

But FD is equal to c by construction, whence c is greater than AK ; or, which is the same thing, AK is less than c , as was to be shewn.

PROP.

10. A cone is a solid generated by the revolution of a right line about the circumference of a circle, one end of which is fixed at a point above the plane of that circle.

11. The axis of a cone is the right line joining the vertex, or fixed point, and the centre of the circle about which the figure is described.

12. Similar cones and cylinders are such as have their altitudes and the diameters of their bases proportional.

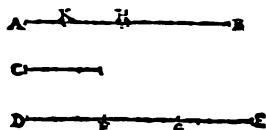
13. A sphere is a solid generated by the revolution of a semi-circle about its diameter, which remains fixed.

14. The axis of a sphere is the right line about which the semi-circle revolves; and the centre is the same as that of the semi-circle.

15. The diameter of a sphere is any right line passing through the centre, and terminated both ways by the surface.

P R O P. I. LEMMA.

If from the greater of two magnitudes, there be taken more than its half; and from the remainder, more than its half; and so on: there will at length remain a magnitude less than the least of the proposed magnitudes.



Let AB and C be any two magnitudes, of which AB is the greater; then, if from AB there be taken more than

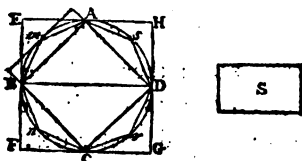
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But the polygon $ABCDE$ is to the polygon $FGHKL$ as the square of BA is to the square of GF (VI. 17.), therefore the polygon $ABCDE$ is also to the polygon $FGHKL$ as the square of BM is to the square of GN .

Q. E. D.

PROP. III. THEOREM.

A polygon may be inscribed in a circle that shall differ from it by less than any assigned magnitude whatever.



Let $ABCD$ be a circle, and s any given magnitude whatever; then may a polygon be inscribed in the circle $ABCD$ that shall differ from it by less than the magnitude s .

For, let AC , EG be two squares, the one described in the circle $ABCD$, and the other about it (IV. 6, 7.); and bisect the arcs AB , BC , CD , DA , in the points m , n , r and s (III. 28.); and join Am , mB , Bn , nC , Cr , rD , Ds and sA :

Then since the square AC is half the square EG (I. 52.), and the square EG is greater than the circle $ABCD$, the square AC will be greater than half the circle $ABCD$.

In like manner, if tangents be drawn to the circle through the points m , n , r , s , and parallelograms be described

scribed upon the right lines AB, BC, CD, DA, the triangles AMB, BNC, CRD, DSA will each of them be half the parallelogram in which it stands (I. 32.)

But every segment is less than the parallelogram which circumscribes it; and therefore each of the triangles AMB, BNC, CRD, DSA is greater than half the segment of the circle which contains it.

And, if each of the arcs Am, mB, &c. be again divided into two equal parts, and right lines be drawn to the points of bisection, the triangles thus formed, may in like manner, be shewn to be greater than half the segments which contain them; and so on continually.

Since, therefore, the circle ABCD is greater than the space s, and from the former there has been taken more than its half, and from the remainder more than its half, &c. there will at length remain segments which, taken together, shall be less than the excess of the circle ABCD above the space s (VIII. 1.), as was to be shewn.

PROP. IV. THEOREM.

A polygon may be circumscribed about a circle that shall differ from it by less than any assigned magnitude whatever.



Let ABCD be the circle, and s any given magnitude whatever; then may a polygon be circumscribed about the

the circle $ABCD$, that shall differ from it by less than the magnitude s .

For let the circle $ABCD$ be circumscribed by the square $EFGH$ (IV. 7.), and bisect the arcs AB , BC , CD , DA with the lines OE , OF , OG and OH ; and to the points of bisection draw the tangents kl , mn , pr , st (III. 10.)

Then since kl is a tangent to the circle, and OE is drawn from the centre through the point of contact, the angle Exk is a right angle (III. 12.); and zk will be greater than kx (I. 17.) or its equal kA .

But triangles of the same altitude are to each other as their bases (VI. 1.); whence the base zk being greater than the base kA , the triangle Exk will also be greater than the triangle kxA .

And because the triangle Exk is greater than the triangle kxA , it will also be greater than half the curvilinear space ExA : and the same may be shewn of any other triangle and the curvilinear space to which it belongs.

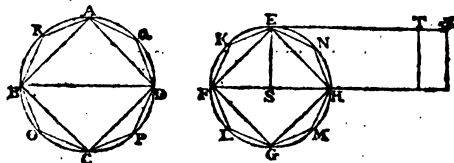
In like manner, if the arcs Ax , xB , &c. be again bisected, and tangents be drawn to the points of bisection, the triangles thus formed will be greater than half the curvilinear spaces to which they belong.

Since, therefore, the excess of the square above the circle is greater than the magnitude s , and from the former there has been taken more than its half, and from the remainder more than its half, and so on, there will at length remain spaces, which, taken together, shall be less than the magnitude s (VIII. 1.), as was to be shewn.

PROP.

PROP. V. THEOREMS.

Circles are to each other as the squares of their diameters.



Let $ABCD$, $EFGH$ be two circles, and BD , FH their diameters: then will the square of BD be to the square of FH as the circle $ABCD$ is to the circle $EFGH$.

For, if they have not this ratio, the square of BD will be to the square of FH , as the circle $ABCD$ is to some space either less or greater than the circle $EFGH$.

First, let it be to a space ST less than the circle $EFGH$; and inscribe the two similar polygons $AROPQ$, $EKLMN$ so that the circle $EFGH$ may exceed the latter by less than it exceeds the space ST (VIII. 3.)

Then, since the circle $EFGH$ exceeds the polygon $EKLMN$ by less than it exceeds the space ST , the polygon $EKLMN$ will be greater than the space ST .

And, because similar polygons, inscribed in circles, are to each other as the squares of their diameters (VIII. 2.), the square of BD will be to the square of FH as the polygon $AROPQ$ is to the polygon $EKLMN$,

But the square of BD is also to the square of FH as the circle $ABCD$ is to the space ST (*by Const.*); whence the circle $ABCD$ will be to the space ST , as the polygon $AROPQ$ is to the polygon $EKLMN$.

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The circle $ABCD$, therefore, being greater than the polygon $AROPQ$, which is contained in it, the space ST will also be greater than the polygon $EKLMN$.

It is, therefore, less and greater at the same time, which is impossible; consequently the square of BD is not to the square of FH as the circle $ABCD$ is to any space less than the circle $EFGH$.

And, in the same manner, it may be demonstrated, that the square of FH is not the square of BD as the circle $EFGH$ is to any space less than the circle $ABCD$.

Nor, is the square of BD to the square of FH as the circle $ABCD$ is to a space greater than the circle $EFGH$.

For, if it be possible, let it be so to the space SX , which is greater than the circle $EFGH$.

Then, since the square of BD is to the square of FH as the circle $ABCD$ is to the space SX , therefore, also, inversely, the square of FH is to the square of BD as the space SX is to the circle $ABCD$ (V. 7.)

But the space SX is greater than the circle $EFGH$ (*by Hyp.*); whence the space SX is to the circle $ABCD$ as the circle $EFGH$ is to some space less than the circle $ABCD$ (V. 14.)

The square of FH is, therefore, to the square of BD as the circle $EFGH$ is to a space less than the circle $ABCD$ (V. 11.), which has been shewn to be impossible.

Since, therefore, the square of BD is not to the square of FH as the circle $ABCD$ is to any space either less or greater than the circle $EFGH$, the square of BD must be to the square of FH as the circle $ABCD$ is to the circle $EFGH$.

Q. E. D.

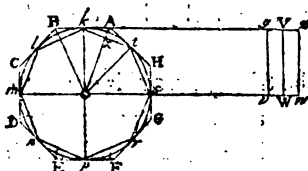
COR. 1. Circles are to each other as the squares of their radii; these being half the diameters.

COR.

COR. 2. If the radii or diameters of three circles be respectively equal to the three sides of a right angled triangle, that whose radius or diameter is the hypotenuse will be equal to the other two taken together (II. 14.)

P R O P. VI. THEOREM.

Every circle is equal to the rectangle of its radius, and a right line equal to half its circumference.



Let $k m p s$ be a circle, and $o v$ a rectangle contained under the radius $o k$ and a right line $o v$ equal to half the circumference; then will the circle $k m p s$ be equal to the rectangle $o v$.

For if it be not, it must be either greater or less.

Let it be greater; and let the rectangle $o x$ be equal to the circle $k m p s$; and inscribe a polygon $i n r t$ in the circle $k m p s$ that shall differ from it by less than the magnitude $w z$ (VIII. 3.)

Then since the triangle $k o t$ is equal to half a rectangle under the base $k t$ and the perpendicular $o x$ (I. 32.), the whole polygon will be equal to half a rectangle under its perimeter and the perpendicular $o x$.

And because $o v$ is greater than half the perimeter of any polygon that can be inscribed in the circle $k m p s$
(by

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(*by Hyp.*), and ok is greater than ox (I. 17.), the rectangle ov will also be greater than the polygon $lnrt$.

But the polygon differs from the circle, or from the rectangle oz , by less than the magnitude wz (*by Const.*), and ov differs from oz by wz ; consequently the polygon is greater than the rectangle ov .

It is, therefore, both greater and less at the same time, which is absurd; whence the circle $kmps$ is not greater than the rectangle ov .

Again, let it be less than ov , by the rectangle wy ; and let $BDFH$ be a polygon circumscribed about the circle, that shall differ from it by less than the magnitude wy (VIII. 4.)

Then since the triangle BOA is equal to half a rectangle under the base BA and the perpendicular ok (I. 32.), the whole polygon will be equal to half a rectangle under its perimeter and perpendicular ok .

And because ow is less than half the perimeter of any polygon that can be circumscribed about the circle (*by Hyp.*), and ok is common, the rectangle ov will also be less than the polygon $BDFH$.

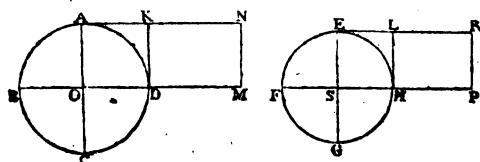
But the polygon differs from the circle, or from oy , by less than the magnitude wy (*by Hyp.*), and ov differs from oy by wy ; consequently the rectangle ov will be greater than the polygon $BDFH$, which is absurd.

Since, therefore, the rectangle ov is neither greater nor less than the circle $kmps$, it must be equal to it, as was to be shewn.

P R O P.

P R O P. VII. THEOREM.

The circumferences of circles are in proportion to each other as their diameters.



Let ABCD, EFGH be any two circles, whose diameters are BD, FH; then will the circumference ABCD be to the circumference EFGH as the diameter BD is to the diameter FH.

For let OM, SP be two right lines equal to the semi-circumferences DAB, HEF, and on the radii OA, SE make the squares OK SL (II. 1.), and complete the rectangles ON, SR:

Then since the rectangles ON, SR are equal to the circles ABCD, EFGH (VIII. 6.), and the circles are to each other as the squares of their radii (VIII. 5. Cor.) the rectangle ON will also be to the square OK as the rectangle SR is to the square SL (V. 9.)

But ON is to OK as OM to OD (VI. 1.), and SR to SL as SP to SH (VI. 1.); therefore, by equality, OM will be to OD as SP is to SH (V. 11.)

And because any equimultiples of four proportional quantities, are also proportional (V. 13.), twice OM will be to twice OD as twice SP is to twice SH; or, by alternation, twice OM is to twice SP as twice OD is to twice SH.

Q

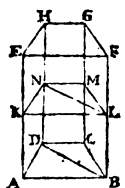
But

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But twice OM and twice SP are equal to the circumferences $ABCD$, $EFGH$ (*by Const.*); and twice OD and twice SH are equal to the diameters BD , FH ; whence the circumference $ABCD$ is to the circumference $EFGH$ as the diameter BD is to the diameter FH . Q. E. D.

P R O P. VIII. THEOREM.

If a prism be cut by a plane parallel to its base, the section will be equal and like the base.



Let AG be a prism, and $KLMN$ a plane parallel to the base $ABCD$; then will $KLMN$ be equal and like $ABCD$.

For join the points NL , and DB :

Then since KM , AC are parallel planes (*by Hyp.*), and the plane AN cuts them, the section KN will be parallel to the section AD (VII. 12.)

And since AK is also parallel to DN (VIII. Def. 3.), the figure AN is a parallelogram; and consequently KN is equal to AD (I. 30.)

In like manner it may also be shewn, that KL is equal to AB , LM to BC , and MN to CD .

And since KN , KL in the plane KM , are parallel to AD , AB in the plane AC , the angle NKL will be equal to the angle DAB (VII. 7.)

The two sides KN , KL of the triangle KLN , being, therefore, equal to the two sides AD , AB of the triangle ABD , and the angle NKL to the angle DAB , the triangle KLN will be equal and like the triangle ABD (I. 4.)

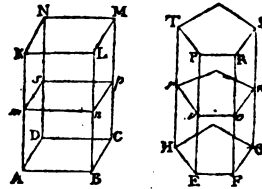
And in the same manner it may be shewn, that the triangle LMN is equal and like to the triangle BCD .

But the triangles KLN , LMN are, together, equal to the section $KLMN$; and the triangles ABD , BCD to the section $ABCD$; whence the section $KLMN$ is equal and like to the section $ABCD$.

Q. E. D.

PROP. IX. THEOREM.

Prisms of equal bases and altitudes are equal to each other.



Let AM , ES be any two prisms, standing upon the equal bases $ABCD$, $EFGH$, and having equal altitudes; then will AM be equal to ES .

For parallel to the bases, and at equal distances from them, draw the planes mp and vw .

Then, by the last proposition, the section mnp will be equal to the base $ABCD$, and the section vw to the base $EFGH$.

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But the base $ABCD$ is equal to the base $EFCH$ by hypothesis; whence the section $mnpq$ is, also, equal to the section $vour$.

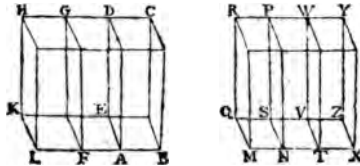
And in the same manner it may be shewn, that any other sections, at equal distances from the bases, are equal to each other.

Since therefore every section in the prism AM is equal to its corresponding section in the prism ES , the prisms themselves, which are composed of those sections, must also be equal. Q. E. D.

COR. Every prism is equal to a rectangular parallelepipedon of an equal base and altitude.

P R O P. X. THEOREM.

Rectangular parallelepipedons, of equal altitudes, are to each other as their bases.



Let AC , MP be two rectangular parallelepipedons, having the equal altitudes ED , QR ; then will AC be to MP as the base BE is to the base NQ .

For in AB , produced, take any number of right lines AF , FL each equal to AB ; and in MN , produced, take any number of right lines NT , TX each equal to MN .

Complete the parallelograms FE , FK , MV , TZ , and make the upright solids AG , FH , NW , TY of equal altitudes with AC or MP .

Then,

Then, because AF , FL are each equal to AB , and NT , TX are each equal to MN (*by Const.*), the parallelograms FE , FK will be each equal to BE , and the parallelograms NV , TZ to NQ (II. 5.)

And, since the solids AG , FH have equal bases and altitudes with the solid AC , they will be each equal to AC (VIII. 9.); and, for the same reason, the solids NW , TY , will be each equal to NR .

Whatever multiple, therefore, the base BK is of the base BE , the same multiple will the solid BH be of the solid AC ; and, for the same reason, whatever multiple the base MZ is of the base NQ , the same multiple will the solid MY be of the solid NR .

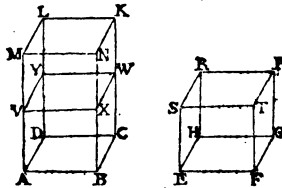
If, therefore, the base BK be equal to the base MZ , the solid BH will be equal to the solid MY ; and if greater, greater; and if less, less; whence the base BE is to the base NQ , as the solid AC is to the solid NR (V. Def. 5.)

Q. E. D.

COR. From this demonstration, and the Cor. to the last Prop. it appears that all prisms of equal altitudes, are to each other as their bases; every prism being equal to a rectangular parallelepipedon of an equal base and altitude.

P R O P. XI. THEOREM.

Rectangular parallelepipeds of equal bases are to each other as their altitudes.



Let AK , EP be two rectangular parallelepipeds standing on the equal bases AC , EG ; then will AK be to EP as the altitude AM is to the altitude ES .

For let AW be a rectangular parallelepiped on the base AC , whose altitude AV is equal to the altitude ES of the parallelepiped EP :

Then, since the base AC is equal to the base EG (*by Hyp.*), and the altitude AV is equal to the altitude ES (*by Conf.*), the solid AW will be equal to the solid EP (VIII. 9.)

And if AL , AY be considered as bases, the solid AK will be to the solid AW as the base AL is to the base AY (VIII. 10.)

But the base AL is to the base AY as the side AM is to the side AV (VI. 1.); whence by equality the solid AK will be to the solid AW as the altitude AM is to the altitude AV (V. 11.)

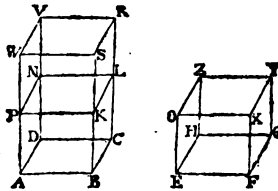
Since, therefore, the solid AW is equal to the solid EP , and the altitude AV to the altitude ES , the solid AK will also be to the solid EP as AM is to ES (V. 9.)

Q. E. D.
COR.

COR. From the reason given in the *Cor.* to the last Prop. it follows, that all prisms of equal bases, are to each other as their altitudes.

P R O P. XII. THEOREM.

The bases and altitudes of equal rectangular parallelepipeds are reciprocally proportional; and if the bases and altitudes be reciprocally proportional, the parallelepipeds will be equal.



Let the rectangular parallelepiped AR be equal to the rectangular parallelepiped EY ; then will the base AC be to the base EG , as the altitude EO is to the altitude AW .

For let AL be a rectangular parallelepiped on the base AC , whose altitude AP is equal to EO , the altitude of the parallelepiped EY .

Then since the altitudes AP , EO are equal to each other (*by Const.*), the solid AL will be to the solid EY as the base AC is to the base EG (VIII. 10.)

And because the solid AR is equal to the solid EY (*by Hyp.*), the solid AL will be to the solid AR as AC is to EG (V. 9.)

Q 4

But

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But the solid AL is to the solid AR as AP is to AW (VIII. 11.); whence, also, AC is to EG as AP is to AW (V. 11.), or AC to EG as EO to AW.

Again, let AC be to EG as EO is to AW; then will AB be equal to EY.

For, since AL is to EY as AC to EG (VIII. 10.), and AC to EG as EO to AW (*by Hyp.*), AL will be to EY as EO to AW (V. 11.)

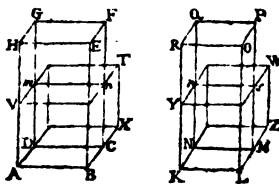
But EO, or AP, is to AW as AL is to AR (VIII. 11.); therefore AL will be to EY as AL is to AR (V. 11.)

And since the antecedents are equal, the consequents will also be equal; whence the solid AR is equal to the solid EY, as was to be shewn.

COR. The same proportion will hold of prisms in general; these being equal to rectangular parallelepipeds of equal bases and altitudes,

P R O P. XIII. THEOREM.

Similar rectangular parallelepipeds are to each other as the cubes of their like sides.



Let AF, KP be two similar rectangular parallelepipeds, whose like sides are AB, KL; then will AF be to KP as the cube of AB is to the cube of KL.

For let AT, KW be two cubes standing on AX, KZ, the squares of the sides AB, KL.

Then

Then since parallelepipeds on the same base are to each other as their altitudes (VIII. 11.), AF will be to $A\pi$ as AH to AV , or AB ; and KP to Ks as KR to KY , or KL .

But the planes $ABEH$, $KLOR$ being similar (VIII. *Def.* 2.), AH will be to AB as KR is to KL (VI. *Def.* 1.); whence AF is to $A\pi$ as KP to Ks (V. 11.); or AF to KP as $A\pi$ to Ks (V. 15.)

Again, since parallelepipeds of the same altitude are to each other as their bases (VIII. 10.), AT will be to $A\pi$ as AX to AC ; and KW to Ks as KZ to KM .

And because AX , or the square of AB , is to AC , as KZ , or the square of KL , is to KM (VI. 17.); AT will be to $A\pi$ as KW is to Ks (V. 11.); or AT to KW as $A\pi$ to Ks (V. 15.)

But AF has been shewn to be to KP as $A\pi$ is to Ks ; therefore, also, AF is to KP as AT to KW (V. 11.)

Q. E. D.

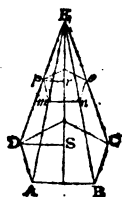
COR. 1. Similar rectangular parallelepipeds are to each other as the cubes of their altitudes; these being considered as like sides of the solids.

COR. 2. Every prism being equal to a parallelepipedon of an equal base and altitude (VIII. 9. *Cor.*), all similar prisms will be to each other as the cubes of their altitudes, or like sides.

PROP.

P R O P. XIV. THEOREM.

If a pyramid be cut by a plane parallel to its base, the section will be to the base as the squares of their distances from the vertex,



Let $EDABC$ be a pyramid, and mn a section parallel to the base AC ; then will mn be to AC as the squares of their distances from the vertex.

For draw es perpendicular to the plane of the base AC (VII. 9.); and join DS and pr .

Then, since mp , mn are parallel to AD , AB (VII. 12.), the angle pnm will be equal to the angle DAB (VII. 7.); and pm will be to DA as Em to EA , or as mn to AB (VI. 3.)

For a like reason each of the angles in the section mn are equal to their corresponding angles in the base AC , and the sides about them are proportional; whence mn is similar to AC (VI. Def. 1.)

And because pm is parallel to DA , and pr to DS (VII. 12.), pm will be to DA as Ep to ED , or as Er to ES (VI. 3.)

The

The lines pm , DA , Er and ES being, therefore, proportional, the square of pm will be to the square of DA , as the square of Er is to the square of ES (VI. 19. *Cor.*)

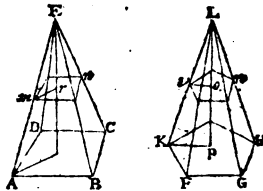
But the square of pm is to the square of DA as mo is to AC (VI. 17.); whence the square of Er is to the square of ES as mo is to AC (V. 11.)

Q. E. D.

COR. If a pyramid be cut by a plane parallel to its base, the section will be similar to the base.

PROP. XV. THEOREM.

Pyramids of equal bases and altitudes are equal to each other.



Let $EDABC$, $LKFGH$ be any two pyramids, of which the base AC is equal to the base FH , and the altitude ES to the altitude LP ; then will $EDABC$ be equal to $LKFGH$.

For make Er equal to Lo ; and draw the sections mn , vw , parallel to the bases AC , FH .

Then, by the last proposition, the square of Er is to the square of ES as mn is to AC ; and the square of Lo to the square of LP as vw is to FH .

And since the square of Er is equal to the square of Lo (*Const. and II. 2.*); and the square of ES to the square of

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of LP (*Hyp. and II. 2.*) mn will be to AC as vw is to FH (V. 9.)

But AC is equal to FH , by hypothesis; whence mn is, also, equal to vw (V. 10.)

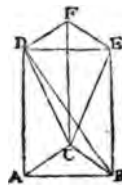
And, in the same manner, it may be shewn, that any other sections, at equal distances from the vertices, are equal to each other.

Since, therefore, every section in the pyramid $EDABC$ is equal to its corresponding section in the pyramid $LKFGH$, the pyramids themselves, which are composed of those sections, must also be equal.

Q. E. D.

P R O P. XVI. THEOREM.

Every pyramid of a triangular base, is the third part of a prism of the same base and altitude.



Let $DABC$ be a pyramid, and $FDABE$ a prism, standing upon the same base ABC , and having the same altitude; then will $DABC$ be a third of $FDABE$.

For in the planes of the three sides of the prism, draw the diagonals DB , DC and CE :

Then

Then because DB divides the parallelogram AE into two equal parts, the pyramid whose base is ABD , and vertex c , is equal to the pyramid whose base is BED and vertex c (VIII. 15.)

And since the opposite ends of the prism are equal to each other (VIII. *Def. 3.*), the pyramid whose base is ABC and vertex D , is equal to the pyramid whose base is DEF and vertex c (VIII. 15.)

But the pyramid whose base is ABC and vertex D , is equal to the pyramid whose base is ABD and vertex c , being both contained by the same planes.

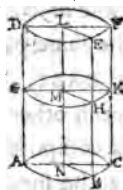
The three pyramids $DABC$, $CBED$ and $CEFD$ are, therefore, all equal to each other; and consequently the prism $FDABE$, which is composed of them, is triple the pyramid $DABC$, as was to be shewn.

COR. Every pyramid is the third part of a prism of the same base and altitude; since the base of the prism, whatever be its figure, may be divided into triangles, and the whole solid into triangular prisms, and pyramids.

SCHOLIUM. Whatever has been demonstrated of the proportionality of prisms, holds equally true of pyramids; the former being always triple the latter.

P R O P. XVII. THEOREM.

If a cylinder be cut by a plane parallel to its base, the section will be a circle, equal to the base.



Let AF be a cylinder, and GHK a section parallel to its base ABC ; then will GHK be a circle, equal to ABC .

For let the planes NE , NF pass through the axis of the cylinder LN , and meet the section GHK in M , H and K .

Then, since the circle DEF is equal and parallel to the circle ABC (VIII. Def. 8.), the radii LF , LE will be equal and parallel to the radii NC , NB (III. 5. and VII. 12.)

And because lines which join the corresponding extremes of equal and parallel lines are themselves parallel (I. 29.), FC , EB will be parallel to LN ; or KC , HB to MN .

In like manner, since the circle GHK is parallel to the circle ABC (*by Hyp.*), MK , MH will be parallel to NC , NB .

And, because the opposite sides of parallelograms are equal (I. 30.), MK will be equal to NC , and MH to NB .

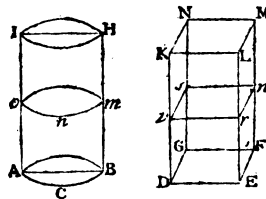
But

But NC , NB are equal to each other, being radii of the same circle; whence MK , MH are also equal to each other.

And the same may be shewn of any other lines, drawn from the point M , to the circumference of the section GHK ; consequently GHK is a circle, and equal to ABC , as was to be shewn.

P R O P. XVIII. THEOREM.

Every cylinder is equal to a prism of an equal base and altitude.



Let AH be a cylinder, and DM a prism, standing upon equal bases ACB , DEF , and having equal altitudes; then will AH be equal to DM .

For parallel to the bases, and at equal distances from them, draw the planes onm , and vrw .

Then by the last Prop. and Prop. 8, the section onm is equal to the base ACB , and the section vrw to the base DEF .

But the base ACB is equal to the base DEF , by hypothesis; whence the section onm is also equal to the section vrw .

And,

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And, in the same manner, it may be shewn, that, any other sections, at equal distances from the base, are equal to each other.

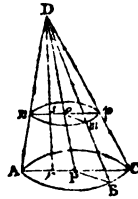
Since, therefore, every section of the cylinder is equal to its correspondent section in the prism, the solids themselves, which are composed of those sections, must also be equal.

Q. E. D.

SCHOLIUM. Whatever has been demonstrated of the proportionality of prisms, holds equally true of cylinders; the former being equal to the latter.

PROP. XIX. THEOREM.

If a cone be cut by a plane parallel to its base, the section will be to the base as the squares of their distances from the vertex.



Let $DABC$ be a cone, and mnp a section parallel to the base ABC ; then will mnp be to ABC as the squares of their distances from the vertex.

For draw the perpendicular Dr ; and let the planes CDP , BDP pass through the axis of the cone, and meet the section in o , p , and m .

Then

Then since the section nmp is parallel to the base ABC (by *Hypo*), and the planes Do , Co cut them, op will be parallel to PC , and om to PB (VII. 12.)

And because the triangles formed by these lines are equiangular, om will be to PB as Do to DP , or as op to PC (VI. 5.)

But PB is equal to PC , being radii of the same circle; wherefore om will also be equal to op (V. 10.)

And the same may be shewn of any other lines drawn from the point o to the circumference of the section nmp ; whence nmp is a circle.

Again, by similar triangles, Ds is to Dr as Do to DP , or as om to PB ; whence the square of Ds is to the square of Dr as the square of om is to the square of PB (VI. 19.)

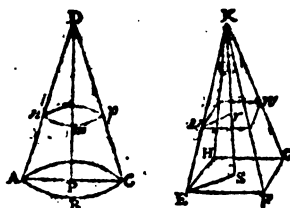
But the square of om is to the square of PB as the circle nmp is to the circle ABC (VIII. 5.); therefore the square of Ds is to the square of Dr as the circle nmp is to the circle ABC (V. 11.)

Q. E. D.

COR. If a cone be cut by a plane parallel to its base the section will be a circle.

PROP. XX. THEOREM.

Every cone is equal to a pyramid of an equal base and altitude.



Let $DABC$ be a cone, and $KEFGH$ a pyramid, standing upon equal bases ABC , $EFGH$, and having equal altitudes DP , KS ; then will $DABC$ be equal to $KEFGH$.

For parallel to the bases, and at equal distances Dn , Kr from the vertices, draw the planes nmp and uvw .

Then by the last Prop. and Prop. 13, the square of Dn is to the square of DP as nmp is to ABC ; and the square of Kr to the square of KS as uvw to EG .

And since the squares of Dn , DP are equal to the squares of Kr , KS (*Const. and II. 2.*), nmp is to ABC as uvw is to EG (*V. 11.*)

But ABC is equal to EG , by hypothesis; wherefore nmp is, also, equal to uvw (*V. 10.*)

And, in the same manner, it may be shewn, that any other sections, at equal distances from the vertices, are equal to each other.

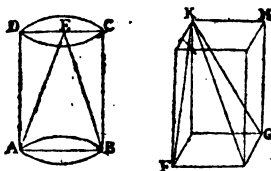
Since, therefore, every section in the cone is equal to its corresponding section in the pyramid, the solids $DABC$, $KEFGH$ of which they are composed, must be equal.

Q. E. D.

PROP.

PROP. XXI. THEOREM.

Every cone is the third part of a cylinder of the same base and altitude.



Let EAB be a cone, and $DABC$ a cylinder, of the same base and altitude; then will EAB be a third of $DABC$.

For let KFG , $KFGH$ be a pyramid and prism, having an equal base and altitude with the cone and cylinder.

Then since cylinders and prisms of equal bases and altitudes are equal to each other (VIII. 18.) the cylinder $DABC$ will be equal to the prism $KFGH$.

And, because cones and pyramids of equal bases and altitudes are equal to each other (VIII. 20.), the cone EAB will be equal to the pyramid KFG .

But the pyramid KFG is a third part of the prism $KFGH$ (VIII. 16.), wherefore the cone EAB is, also, a third part of the cylinder $DABC$.

Q. E. D.

SCHOLIUM 1. Whatever has been demonstrated of the proportionality of pyramids, prisms, or cylinders, holds equally true of cones, these being a third of the latter.

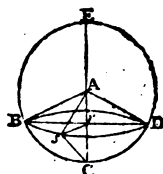
2. It is also to be observed, that similar cones and cylinders are to each other as the cubes of their altitudes,

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or the diameters of their bases; the term like sides being here inapplicable.

PROP. XXII. THEOREM.

If a sphere be cut by a plane the section will be a circle.



Let the sphere EBD be cut by the plane BSD; then will BSD be a circle.

For let the planes ABC, ASC pass through the axis of the sphere EC, and be perpendicular to the plane BSD.

Also draw the line BD; and join the points A, D and r, s:

Then since each of these planes are perpendicular to the plane BSD, their common section Ar will also be perpendicular to that plane (VII. 14.)

And, because the sides AB, Ar, of the triangle ABr are equal to the sides As, Ar of the triangle Asr, and the angles ArB, Ars are right angles, the side rB will be equal to the side rs (I. 4.)

In like manner, the sides AD, Ar, of the triangle ADr, being equal to the sides As, Ar, of the triangle Asr, and the angles ArD, Ars right angles, the side rD will also be equal to the side rs (I. 4.)

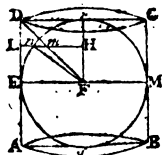
The lines rB, rD and rs, are, therefore, all equal; and the same may be shewn of any other lines, drawn from the point

point r to the circumference of the section; whence BSr is a circle, as was to be shewn.

COR. The centre of every section of a sphere is always in a diameter of the sphere.

PROP. XXIII. THEOREM.

Every sphere is two thirds of its circumscribing cylinder.



Let $rESM$ be a sphere, and $DABC$ its circumscribing cylinder; then will $rESM$ be two thirds of $DABC$.

For let AC be a section of the sphere through its centre F ; and parallel to DC , or AB , the base of the cylinder, draw the plane LH , cutting the former in n and m ; and join FE , Fn , FD and Fr .

Then, if the square Er be conceived to revolve round the fixed axis Fr , it will generate the cylinder EC ; the quadrant FEr will also generate the hemisphere $EMrE$; and the triangle FDr the cone FDC .

And since FHn is a right angled triangle, and FH is equal to Hm , the squares of FH , Hn , or of Hm , Hn , are equal to the square of Fn .

But Fn is also equal to FE or HL ; whence the squares of Hm , Hn are equal to the square of HL : or the circular

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sections whose radii are Hm , Hn are equal to the circular section whose radius is HL (VIII. 5. *Cor.*)

And as this is always the case, in every parallel position of HL , the cone FDC and cylinder EC , which are composed of the former of these sections, are equal to the hemisphere $EMrE$, which is composed of the latter.

But the cone FDC is a third part of the cylinder EC (VIII. 21.); whence the hemisphere $EMrE$ is equal to the remaining two thirds; or the whole sphere $rESM$ is equal to two thirds of the whole cylinder $DABC$, as was to be shewn.

COR. 1. A cone, hemisphere, and cylinder, of the same base and altitude, are to each other as the numbers 1, 2, and 3.

COR. 2. All spheres are to each other as the cubes of their diameters; these being like parts of their circumscribing cylinders.

NOTES AND OBSERVATIONS.

DEF. 1. BOOK I.

THE definition of a solid, contrary to the usual method, is here made the first of the first Book; as those of a point, line and superficies are all derived from it, and cannot be understood without it. EUCLID seems to have placed it in the eleventh book of the Elements, for the sake of uniformity; but arrangements of this kind, which are merely arbitrary, are but of little consequence, and should therefore always be made to give place to perspicuity and the natural order of things.

DEF. 2, 3, 4. BOOK I.

These definitions are now, by means of the former, rendered perfectly clear and intelligible, so that any farther elucidation of them is altogether unnecessary. DR. SIMSON has endeavoured to shew, by a formal proof, drawn from the consideration of a solid, that a point, according to EUCLID'S definition, is without parts, a line without breadth, and a surface without thickness; but this, and all other demonstrations of the same kind, are unscientific and superfluous; for these properties are so obviously essential to the things defined, that they cannot, even in idea, be separated from them. If a point had parts, it would be a line; if a line had breadth it would be a superficies; and if a superficies had thickness, it would be a solid; which are all manifest contradictions. It is, besides, a sure sign that a definition is badly expressed,

when it requires a number of prolix arguments to establish its truth and propriety.

DEF. 5. BOOK I.

EUCLID's definition of a right line is not expressed in so accurate and scientific a manner as could be wished; the *lying evenly* between its extreme points, is too vague and indefinite a term to be used in a science so much celebrated for its strictness and simplicity as Geometry. **ARCHIMEDES** defines it to be the shortest distance between any two points; but this is equally exceptionable, on account of the uncertain signification of the word distance, which, in common language, admits of various meanings. That which is here given, is, perhaps, not much preferable to either of these. The term right, or straight line, is, indeed, so common and simple, that it seems to convey its own meaning, in a more clear and satisfactory manner than any explanation which can be given of it. **DR. AUSTIN**, in his Examination of the first six books of the Elements, proposes a singular emendation of this definition, which includes the consideration of right lines, instead of a right line, as the case manifestly requires.

DEF. 6. BOOK I.

Some call a plane superficies that which is the least of all those having the same bounds: and others, that which is generated by the motion of a right line, not moving in the direction of itself; but these definitions are too complex and obscure to answer the purpose required. **EUCLID** defines it to be that which *lies evenly* between its lines; which is liable to the same exception as that given of a right line: nor is the one which has been substituted in

in the place of this, by DR. SIMSON, and other Editors, so simple and perspicuous as could be wished. Nothing is gained by the explanation of a term, if the words in which it is expressed are equally, or more, ambiguous, than the term itself: for this reason, that which is here given, has been preferred to either of those abovementioned: though, perhaps, it may not be equally commodious in certain cases.

It is also to be remarked, that EUCLID never defines one thing by the intervention of another, as is the case in DR. SIMSON's emendation; so that if this method had occurred to him, he would certainly have rejected it.

DEF. 7. BOOK I.

The general definition of an angle in EUCLID, has been properly objected to, by several of the modern Editors, as being unnecessary, and conveying no distinct meaning; and in DR. SIMSON's emendation of the ninth, there seems to be still a superfluous condition. He defines a rectilinear angle, to be "the inclination of two straight lines to one another, which meet together, but are not in the same straight line." Now their not being in the same straight line, is a necessary consequence, obviously included in their having an inclination to each other; and, therefore, to make this an essential part of the definition, is certainly improper, and unscientific.

DEF. 8, 9. BOOK I.

EUCLID includes a right angle and a perpendicular in the same definition, which appears to be immethodical, and contrary to his usual custom. They are certainly distinct things, though dependent upon each other, and
have

have as much claim to be separately defined, as a circle and its diameter.

DEF. 13. BOOK I.

The definition of a circle from its generation, has been thought by DR. BARROW and others, to be preferable to EUCLID's, or the one here given; as it is supposed to furnish its properties more readily, and to have the still farther advantage of shewing the actual existence of such a figure, independent of any hypothesis, but that of granting the possibility of motion. But the requisition of this postulatum, appears to be a sufficient reason why EUCLID rejected such a definition. The principles of pure Geometry, have no dependence upon motion, and it is, therefore, never used in the Elements, but in two or three places of the eleventh book, where it could not, without much obscurity and circumlocution, have been easily avoided. It is besides, neither so simple, nor convenient to refer to, as EUCLID's; which, in these respects, is as commodious as could be wished.

DEF. 20. BOOK I.

DR. BARROW, and other writers of considerable eminence, have censured EUCLID for defining parallel lines, from the negative property of their never meeting each other; and to this they attribute all the perplexity and confusion, which has hitherto attended this delicate subject: affirming it as an utter impossibility, that any of the properties of these lines, can be derived from a definition which contains only a simple negation. But these assertions appear to be groundless; for the definition is founded on one of the most familiar, simple and obvious properties

properties of parallel lines, which either reason or science can discover: and, on this account, it is certainly preferable to any other that could have been formed from more abstruse and complicated affections of those lines, how ready and useful soever such a definition might have been found in its application.

The assertion, likewise, that none of the other properties of parallel lines can be derived from this definition, has been unadvisedly made; for the 27th Prop. of the first Element, which is the same as the 22d of the present performance, is fairly and elegantly demonstrated by it; and by means something similar to those made use of by DR. SIMSON, in his Notes upon the 29th Prop. it would not be difficult to shew that all the other properties of those lines may be derived from this definition, without the assistance of the 12th axiom, or any other of the same kind. DR. SIMSON, indeed, in his attempt to demonstrate this axiom, has made several paralogisms which render his reasonings altogether invalid, and nugatory. Passing by others, of less consequence, it will be sufficient to observe, that in his fifth Prop. he takes it for granted, that a line, which is perpendicular to one of two parallel lines, may be produced till it meets the other: now this is a particular case of the very thing he is endeavouring to prove, which is so strange an oversight, that it is remarkable how it could escape his observation.

This, however, is not the only instance of an unsuccessful attempt to prove the truth of the 12th axiom; for CLAVIUS and others have committed similar mistakes, and DR. AUSTIN, who has endeavoured to demonstrate it by means of a new definition of parallel lines, has made

made use of an assumption equally unwarrantable with that mentioned above. That the theory of parallel lines, as it is given in the Elements, is very imperfect, cannot be denied; but no one has yet been substituted in its place which is not equally defective; and in some instances still more exceptionable: particularly as they are founded on a definition which is derived from an adventitious property of those lines, instead of one which is inherent and necessary, as the nature of the subject requires.

Whether EUCLID was the author of this axiom cannot perhaps, at this time, be easily determined; but it is certainly a disgrace to the Elements. The truth of the property here assumed as a thing to be granted, is so far from being obvious, that it requires demonstration as much as any Prop. in the Elements; and it is always observed that learners, instead of giving that ready assent to it which an axiomatical principle requires, receive it with doubt and hesitation, and are scarcely able to comprehend the meaning of it. The one which is here made the 4th postulate, though nearly the same thing in effect, is much more clear and intelligible.

PROP. I. BOOK I.

In the demonstration of this proposition, by EUCLID, that part which relates to the intersection of the circles is, very improperly, omitted; for in a work of this kind, nothing, however evident, ought to be taken for granted; and particularly at the first outset, where a strictness of elucidation is peculiarly necessary. The passing of the circles through each other's centres is, indeed, a sufficient reason why they must cut each other; but this should certainly have been mentioned in the demonstration.

PROP.

PROP. 2. BOOK I.

PROCLUS, and other writers, have observed, that this problem admits of several cases, according to the situation of the point A; but there is only one of them that can properly be called a separate case, which is when the point A is at either of the extremities of the given line: and in this case, if a circle be described from the given point, at the distance CB, any of the radii of that circle will be the line required. In all other situations of the point A, whether in the line AB, or out of it, the construction and demonstration will be the same as that given in the text; which differs from EUCLID's only in the producing of the line DA, after the circle FHC is described; this being thought more conformable to the terms of the proposition.

PROP. 3. BOOK I.

In the construction of this problem the line AD may fall upon the line AB, and then the thing required is done. The given lines C and AB may also meet each other, at the point A; and then a circle described from that point, with the radius C, will cut off from AB the part required. This case occurs in the construction of the 5th proposition following, and in several other parts of the Elements, and, for that reason, ought to have been mentioned. In all other positions of the two given lines EUCLID's construction and demonstration are general.

PROP. 4. BOOK I.

The demonstration of this proposition has been frequently objected against, as being too mechanical. But this complaint is frivolous and ill founded; for the operation

ration of placing one triangle upon the other, is a mental one, and what is to be considered as possible to be effected, rather than actually done. There is, besides, no other way in which the equality of these figures can be established, so that any cavils about its merits or defects are entirely precluded.

PROP. 5. BOOK I.

EUCLID, in his demonstration of this proposition, has shewn that the angles below the base are, also, equal to each other. But as this property is never referred to throughout the Elements, except in the demonstration of the 2d case of the 7th proposition following, the whole of which is both awkward and unnecessary, it would have been better to have omitted it, and confined the demonstration, in the present instance, to the equality of the angles above the base, which is a property much more generally useful.

PROP. 6. BOOK I.

The demonstration of this proposition, in EUCLID, is immethodical and defective. It is not sufficient to shew that one side is not greater than the other, but it ought, also, to be shewn that it is not less, before their equality can be fairly inferred. It is true, indeed, that either of the sides may be taken at pleasure, and the same thing will follow: but this observation should have been made, and then the premises, which they do not at present, would have authorized the deduction required. The same objection may be made to several other propositions in the Elements.

PROP.

PROP. 7. BOOK I.

This is the same as EUCLID's 8th proposition; but demonstrated in a different manner, in order that the preceding one, which is altogether useless, might be omitted. PROCLUS demonstrates it in nearly the same manner; but he makes three cases of it, when it may be done generally in one; for if the longest sides, or rather those which are not shorter than any other, be applied together, there can be no ambiguity in the species of the triangles.

PROP. 8. BOOK I.

This proposition is made an axiom by EUCLID; but it is certainly not a truth of that kind: for when two right angles are found in separate and distinct figures, there is nothing in the definitions or postulates, from which their equality to each other can be fairly inferred.

PROP. 12. BOOK I. [⊙]

It is not shewn by EUCLID, in his demonstration of this problem, that the circle made use of in the construction, will cut the given line in two points, which as much requires proof as Prop. 2. Book III. which is nearly its converse. For this reason the construction given in the text has been preferred; but in a work where the utmost scientific rigour is required, it would be better to construct the problem without the intervention of the circle, by means of right lines only, which may easily be done.

PROP. 13. BOOK I.

In the enunciation of this Prop. by EUCLID, the angles are said to be either equal to two right angles, or together

ther equal to two right angles : but the former part of this seems to be unnecessary ; for in all cases, whether the angles be each of them a right angle, or not, they are together equal to two right angles.

PROP. 16. BOOK I.

As the outward angle of a triangle is afterwards shewn to be equal to the two inward opposite angles, it is much to be wished that the present Prop. which is only a partial case of the former, could be removed from the Elements : but this cannot easily be done ; for the following proposition, and the first relating to parallel lines, are not otherwise to be demonstrated. The next Prop. in EUCLID, is, however, quite unnecessary, as the first place in which it occurs is Prop. 18, B. 3, where a reference may be as readily made to the general proposition.

● PROP. 19. BOOK I.

The demonstration of this proposition, as it is given by EUCLID, is extremely defective ; for the whole design of the problem is to shew that of three right lines, under certain specified restrictions, a triangle may be formed ; and as no use whatever is made of these restrictions, either in the construction or demonstration, both the arguments adduced, and the conclusions derived from them, are entirely nugatory. This defect was observed by MR. SIMPSON, between whom and his antagonist DR. SIMPSON, it occasioned some controversy, which drew from the latter some very hasty unscientific expressions, not much comporting with the character of so strict and accurate a Geometrician.

PROP.

PROP. 28. BOOK I.

In most editions of EUCLID, two corollaries are affixed to this proposition; which are equally or more intricate than the proposition itself. DR. AUSTIN has endeavoured to prove that these, and most of the other corollaries, to be found in the Elements, were not introduced by EUCLID, but by some of his commentators; or interpreters; and there are many reasons for believing that this opinion, is not ill founded. It is generally allowed, that EUCLID, wrote a book entitled Corollaries, which were a collection, of consequences deducible from his Elements; and, therefore, it is not to be imagined that they were originally inserted in that work; as in that case it would have been quite unnecessary to have published them in a separate performance. Besides this, the chain of reasoning is complete without them, as is evident from their being seldom referred to in any proposition. In all cases, however, where a useful truth of this kind can be readily deduced from a preceding demonstration, there appears to be no impropriety in making it a corollary.

PROP. 31. BOOK I.

This proposition, as it stands in most of the early editions, has three distinct cases, which all require to be separately demonstrated. DR. SIMSON, by changing the mode of demonstration, has reduced it to two: but by an obvious alteration in the enunciation of the I. 26. EUC. which is the same as our 21st, the proposition, both for parallelograms and triangles, might have been demonstrated generally, in one case only; which, when it can

be done, is always to be preferred. In this part of the work, also, several other alterations have been made, the reason for which will be given in the notes to the second book.

PROP. 33. BOOK I.

This proposition is substituted in the place of Prop. 42, 44, and 45 of EUC. B. I. as being less intricate, and equally useful in its application. One of the principal designs of these propositions, is to shew, that a parallelogram, under certain conditions, can be formed; and as this can be more readily effected by other methods, the preference has been given to that which appears the most simple. It may also be observed, that the 44th proposition of EUCLID is not legally demonstrated; for the parallelogram *BF*, which makes a part of the construction, cannot be formed from Prop. 42, as is directed, being entirely a different case: and as the 45th is derived from the 44th, it must also be liable to the same objection.

PROP. 34, 35. BOOK I.

These propositions are delivered by EUCLID in a different form, and not given till the 6th book; but as they are extremely easy, and of frequent use in their application to other propositions, in the preceding books, they have been here introduced as early as possible, and demonstrated independently of the doctrine of proportion; which, it is imagined, beginners will consider as an advantage, as they seldom arrive to such a proficiency, in a knowledge of the Elements, as to obtain clear and satisfactory ideas of that intricate subject.

PROP.

PROP. 1. BOOK II.

In EUCLID's demonstration of this problem, it ought to have been proved that the lines which are directed to be drawn parallel to AB, AD, will meet each other; or otherwise it is not certain that the square required can be formed. On this account, a mode of construction has been here observed, which is not liable to that objection.

PROP. 2, 3, 4. BOOK II.

These propositions, which are not in EUCLID, may by some, be thought unnecessary; but they must either be demonstrated, or assumed; as the first, in particular, is wanted in almost every proposition of the second book; and the others are frequently required in several parts of the Elements. Why they were omitted by EUCLID does not appear; they are certainly not axiomatical, nor more evident in themselves than many others which he has scrupulously demonstrated.

PROP. 5. BOOK II.

This proposition is placed in the second book, for the purpose of demonstrating it in a more general manner than has been done by EUCLID; and in order that some others, of little importance, might be more easily omitted. The demonstration depends principally upon the first proposition, mentioned above; and this, among other instances, is sufficient to shew the utility of that theorem, and the necessity of its being introduced into the Elements.

PROP. 7. BOOK II.

This theorem, which is not in EUCLID, is given chiefly on account of its application to some of the following propositions; the demonstrations of which are, by this means, rendered more concise and elegant.

E. 2.

PROP. 13. BOOK II.

All the theorems in EUCLID's second book, which relate to the division of a line into more than two parts, are here omitted, as they are commonly found tedious and embarrassing to beginners, and are not of any very extensive use. The present proposition, which is not in EUCLID, is much more generally applicable; and this, together with the preceding ones, will be found sufficient for most geometrical purposes.

PROP. 16, 19, 20 and 21. BOOK II.

These propositions, though not in EUCLID, are frequently wanted, particularly the 1st, 2d, and 3d, which are, also, equally remarkable, both for their elegance and utility.

PROP. 1. BOOK III.

It is properly observed by DR. SIMSON, in his notes upon this book, that the objections which have been usually made against the indirect method of proof, used in this and several other propositions in the Elements, are injudicious and ill founded; as it is obvious to every one, who has duly considered the subject, that there are many things which cannot be proved in any other

other way. There is, however, a real defect in the demonstration of this proposition, that escaped his notice; which is, that the fictitious centre, or point C, may be taken in the line EC; and in this case the demonstration given by EUCLID will not hold.

PROP. 4. BOOK III.

This proposition is the same as the 9th of EUCLID, Book III. but demonstrated in a manner which it is imagined will appear something more clear and satisfactory, at least to beginners. According to his method the proposition admits of several cases; and in that which he has chosen as a general one, the fictitious centre, or point E, is so taken, that the proof would be exactly the same for two equal right lines as for three, which is a manifest imperfection,

PROP. 5. BOOK III.

EUCLID has given this theorem in his 3d Book, in the form of a definition; which is the more remarkable, as he appears, in several parts of the Elements, to be well aware, that the equality of no two figures can be admitted but from the test which he has laid down in the 8th axiom,

PROP. 6, 7. BOOK III.

These theorems are, in substance, the same as EUCLID's, but differently enunciated, in order to accommodate beginners, who are generally embarrassed with the awkwardness of the figures, and the two fictitious centres in the last proposition; the latter of which are

here avoided. It may also be observed that the demonstrations of these propositions are not strictly scientific; since, for aught that appears to the contrary, the circles may touch each other in more points than one, in which case the proof he has given would be nugatory. To avoid this, the succeeding propositions should have been placed prior in order to the present ones, and demonstrated independently of them; which, however, cannot easily be done. For this reason, and to avoid as much as possible all theorems which are otherwise of little importance, the touching of the circles in one point only, has been here inferred from the definition. A similar objection may, likewise, be made against the 5th, 6th, and 10th theorems of EUCLID, Book III; the last of which should have been demonstrated first; for, as they now stand, several things which require proof, as much as the propositions themselves, are taken for granted.

PROP. 10. BOOK III.

In this proposition, no mention is made of the conical angle, or that which is supposed to be formed by the circumference and tangent, at the point of contact; as it is of no use whatever in Geometry, and ought never to have been admitted into the Elements. DR. SIMSON suspects, with VIETA, that it is an interpolation, and on that account has properly rejected it; but there are still some particulars, in his enunciation of this proposition, which appear to be equally unnecessary. The theorem is, therefore, here proposed in as simple a manner as possible, and restricted to that case which most frequently occurs.

PROP.

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PROP. 14. BOOK III.

IN EUCLID's demonstration of the second case of this theorem, the following proposition has been taken for granted, viz. " If one magnitude be double of another, and a part taken from the first, be double of a part taken from the second, the remainder of the first will be double the remainder of the second." But as this assumption, which has hitherto been tacitly acquiesced in, is not derived from the axioms, or any thing which has been previously demonstrated, it is certainly improper, and unjustifiable. In order, therefore, to render the demonstration of this case more strict and scientific, it is here given in a different manner, which is equally easy with the former, and not liable to the same objection.

PROP. 15. BOOK III.

DR. AUSTIN in his examination of the first six books of the Elements, is of opinion that the second case of this proposition, which has been added by DR. SIMPSON, and other modern Editors, is unnecessary. " The former proposition, he observes, is general ; and, therefore, it is immaterial, whether the part of the circumference upon which the angles at the centre and circumference stand, be greater or less than a semicircle." But this observation is foreign to the purpose ; for as the arc which subtends an angle at the centre, must always be less than a semicircle, no such angle can be introduced into the construction of this case ; and therefore the demonstration of it must necessarily be obtained in some way different from the former,

PROP. 18, 19. BOOK III.

The first of these propositions is the same in effect as the 25th of EUCLID, Book III, but something more simple, being demonstrated generally in one case. The other is the converse of our 17th, or EUCLID's 22d, which he has omitted, but for what reason does not appear, as it is of frequent use in its application.

PROP. 20. BOOK III.

This proposition differs from the 24th of EUCLID, Book III, only in the enunciation, which was done to avoid the necessity of defining similar segments of circles. For as this definition, which is nothing more than EUCLID's 21st proposition, in another form, cannot possibly be understood, till it is shewn that all angles in the same segment are equal to each other, it is altogether useless, and contrary to the nature of a definition, which requires that it should be expressed in such terms, and derived from such properties as are simple and obvious.

PROP. 27, 28, 29. BOOK III.

The demonstrations of these propositions are confined to one case, which, though it does not include every possible position of the lines, will, it is conceived, be thought sufficiently general. EUCLID, in this instance, is much more particular; having scrupulously demonstrated cases of less moment than many others in the Elements which are taken for granted. And the same want of uniformity may be observed in several other propositions, with respect to the strictness or laxity of their demonstrations.

PROP.

PROP. 3. BOOK IV.

DR. SIMSON in his note upon EUCLID, Prop. 5, B. 4, observes " that the demonstration of this problem, has been spoiled by some unskilful hand. For he does not demonstrate, as is necessary, that the two straight lines, which bisect the sides of the triangle, at right angles, must meet one another." After which, it appears something singular, that this able Geometrician should not perceive that a similar omission had been made in the demonstration of the 3d, and several other propositions of this book; in which the necessity of proving that certain lines will meet each other is equally obvious. In all these cases, therefore, that part of the demonstration is now supplied, and the different solutions, by that means, rendered more complete.

PROP. 10. BOOK IV.

This proposition has been purposely altered from EUCLID, in order to render the construction of the following one more practical and simple. It is now, also, properly limited, which EUCLID's is not; for according to his enunciation of the problem, an infinite number of triangles may be formed, which will answer the conditions required. The same objection is likewise applicable to several other propositions in the Elements; and though it may, to some, appear trifling, it is certainly a departure from that strictness and precision which, in a work of this nature, are generally considered as essential requisites. An instance of this kind occurs even in the 2d Prop. of B. I, which, however, is not so easily remedied.

PROP.

PROP. 11. BOOK IV.

Some of the Commentators have observed, that EUCLID's method of inscribing a pentagon in a circle, is much less simple and elegant than that of PTOLEMY in the 1st Book of his *Almagest*; and for that reason think it ought to have been given in the *Elements*. Whilst others maintain that the demonstration of PTOLEMY's construction depends wholly on the 13th Book, and that, therefore, if EUCLID had known it, he could not have inserted it in the present Book; the materials for it not being yet prepared. This, however, is not true; for it has been clearly shewn by the author, in a periodical publication for the year 1786, that the truth of this construction may be proved by means of the first three Books of the *Elements* only; but the reason why it has not been given in the text is, that the demonstration, being something more intricate, might not have been so readily comprehended by beginners, for whose use this work is principally designed.

DEF. 5. BOOK V.

This definition, which is the same in effect as that given by EUCLID, has been the occasion of much controversy and dispute among Mathematicians; many of them thinking it foreign to the purpose, and others too difficult and obscure to be made the leading principle of a doctrine so useful and necessary as that of proportion. But, from a mature consideration of the subject, it is sufficiently evident that no other definition, equally applicable and general, could have been given; and, therefore, the necessity

necessity of the case required that the present one should be adopted in preference to all others.

That it is not so simple and evident as that which may be given of numbers, or commensurable magnitudes, cannot be denied; but this arises from the nature of the subject, and is not to be avoided. There are some proportional magnitudes, such, for instance, as the side of a square and its diagonal, which have no common measure, and consequently cannot be defined by that means. Some other definition was, therefore, to be found, which would hold in this, and all other cases, without exception; and as the one in question answers these conditions, and is, at the same time, equally commodious in practice, nothing farther can be expected.

In order, however, to accommodate learners, who are seldom able to comprehend EUCLID's 5th Book, such alterations have been made in this part of the work, as it is presumed will render it much more clear and intelligible. Among others, the definition above-mentioned is more concisely enunciated; by which it is made to appear less intricate and involved; and consequently may be more easily remembered and applied. Every thing which relates to a greater and less ratio is also rejected, as being obscure and unnecessary. And as brevity was here thought particularly requisite, such propositions only have been introduced, as are obviously useful; the rest being considered as impediments in the way of the learner, and, for that reason, unfit for an elementary performance, whose principal aim should be clearness and perspicuity.

For a further account of the doctrine of ratios, as delivered by EUCLID, the reader is referred to the 7th and 8th

8th of DR. BARROW's Mathematical Lectures, where the usual objections which are made to this method are fully refuted.

PROP. 2. BOOK V.

This Prop. is the same as the Cor. to EUCLID's 2d Prop. of Book V, which DR. SIMSON has marked with inverted commas, as being unnecessary. But, whoever considers this Book with attention, will observe, that the corollary is much more useful and general than the proposition. It is, indeed, strictly speaking, no corollary to the proposition in question, and for that reason is properly enough discarded; but it would have been much better to have struck out the proposition, and substituted the corollary in its place.

PROP. 4. BOOK V.

This proposition, which is required in the demonstration of some of the following ones, is not expressly enunciated by EUCLID, being introduced into the demonstration of the 8th proposition, without any farther notice. But as it is a distinct theorem, the truth of which is much less obvious than that of several others in this Book, it ought to have been separately demonstrated; and particularly as it is of considerable use in its application.

PROP. 16, 17. BOOK V.

It has been properly observed, by MR. SIMPSON, that the manner in which the composition and division of ratios is treated of by EUCLID, is defective, as not being sufficiently general. It is also commonly found very
abstruse

abstruse and embarrassing to beginners, on account of the complicated terms in which it is enunciated, and the number of cases to be separately demonstrated. For these reasons; it was deemed necessary to give the propositions a more simple and general form, and to render the demonstrations of them as concise and perspicuous as possible. The 17th, in strictness, has two distinct cases; but as the second differs from the first only by inverting the terms, it was judged sufficient to mention it in a Scholium.

DEF. 1. BOOK VI.

DR. AUSTIN objects to this definition, because it does not yet appear that any rectilinear figures can have their angles equal, each to each, and the sides about them proportional. He would therefore define similar triangles first, which may be done only from the equality of their angles; and after this to call those similar rectilinear figures, of more than three sides, which consist of an equal number of similar triangles, similarly situated. This is no doubt an alteration which might prove advantageous to learners; but as it appears illogical, and contrary to the method made use of in other cases of a like nature, the original definition was thought preferable.

PROP. 12, 13, 16, 17. BOOK VI.

Several alterations have been made in the demonstrations of these propositions, as they were given by EUCLID, with the view of rendering them more simple and easy; but as they are in general such as may be readily discovered by inspection, or by comparing the theorems with those in the Elements, it will be unnecessary to point them out to the reader.

PROP.

PROP. 19, 20, 22, 23. BOOK VI.

These propositions are not in EUCLID, though their utility and elegance certainly entitled them to a place in the Elements. The latter, in particular, may supply the place of the 27th, 28th, and 29th of EUCLID; which are certainly very awkward propositions, and are seldom well understood by beginners. The use which was made of them by some of the ancient Geometers, has been urged as a sufficient reason why they ought to be retained in the Elements; but, upon this principle, numberless other propositions might be inserted, which would swell this compendious and beautiful system into a large volume; and make it appear more like a common place book than a simple regular performance, judiciously arranged in all its parts, and displaying only the first and most important principles of the science.

PROP. 25, 26, 27. BOOK VI.

These elegant and useful theorems were not originally in EUCLID; but have been inserted in some of the late editions, by DR. SIMSON and other writers. MR. SIMPSON has given them in the third book of his Elements, and demonstrated them independently of proportion; which he conceives to be an alteration much for the better: but this will scarcely be allowed, when it is considered that he was obliged to introduce a new theorem for this purpose, manifestly derived from the principles laid down in the 5th Book; and that the advantages attending this method, are entirely destroyed by a forced and unnatural arrangement.

PROP.

PROP. 2. BOOK VII.

DR. SIMSON, in his notes upon this proposition, observes, that the enunciation of it, in most editions, has been changed and vitiated, by its being proposed to shew that every part of a triangle is in the same plane, instead of the way in which it now stands; as the property here alluded to is said to be contained in the definition of a triangle, and is constantly taken for granted in every part of the first six books. But he did not perceive that a similar objection might be made to the preceding proposition; in which it is proved that one part of a right line cannot be in a plane, and another part above it; this being, in like manner, a necessary consequence of the definition he has given of a plane, or rather the very property from which it is derived.

PROP. 3. BOOK VII.

The demonstration of this theorem, which is new, and more concise than that given by EUCLID, is derived from a property of the isosceles triangle, not mentioned in the Elements, but which is found of considerable use in its application to other propositions. Several other alterations have also been made in different parts of this book, of which, as they will be readily discovered by those who are versed in the subject, any farther account is unnecessary.

BOOK VIII.

The method here followed, of treating the solids, is not materially different from that which has been employed
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by other writers upon the like occasion; but the propositions, it is presumed, will be found much better arranged, and the subject rendered more easy and familiar than has hitherto been done. The author however, is well aware that it is liable to critical objections; and had it not been incompatible with the plan of the performance, he certainly would have given it a more scientific form; which even from the simple principles here employed, might easily have been done. But as the advantages attending the present mode of demonstration, especially to learners, appeared superior to every other consideration, it was adapted in preference to that of EUCLID; which, though more accurate, is frequently found to be tedious and obscure.

FINIS.



Published by the same AUTHOR.

1. The SCHOLAR'S GUIDE to ARITHMETIC, 7th Edit.
1802. Price 2s. 6d.

2. AN INTRODUCTION to MENSURATION and PRACTICAL GEOMETRY, 7th Ed. Pr. 4s.

3. AN INTRODUCTION to ALGEBRA, 6th Ed. Pr. 3s. 6d.

4. AN INTRODUCTION to ASTRONOMY, 4th Ed. Pr. 8s. in boards.

Printed for J. JOHNSON, No. 72, St. Paul's Church-Yard.

Bye and Law, Printers, St. John's Square, Clerkenwell.

For the purpose of this study, the data were collected from the following sources:

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